FISCAL RULES AND LONG-TERM SOVEREIGN DEBT: THE CONSEQUENCES OF THE LIFETIME-LAFFER CURVE *†

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I show that long-term sovereign debt can imply equilibrium multiplicity for a fixed fiscal rule. Varying expectations regarding limiting debt levels dilute current prices differently; this price variation alters borrowing, placing the sovereign on the anticipated trajectory. The model assumes the ability to commit to current but not future debt issuance. This implies that equilibria are ranked, which has robust policy implications. I apply the model to the Eurozone crisis and find a counterfactual equilibrium could imply a peak Portuguese spread as much as 15 percentage points (91%) lower and that negative long-run expectations can also generate indeterminacy.

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1 Introduction

Quantitative models of sovereign debt and default have proven quite useful in explaining many key features of emerging market economies (Eaton and Gersovitz [1981], Aguiar and Gopinath [2006], Arellano [2008]). The inclusion of long-maturity debt in these models often helps to improve their fit by generating more reasonable spreads and spread volatility (Hatchondo and Martinez [2009], Chatterjee and Eyigungor [2012], Aguiar et al. [2016]). The theoretical implications of long-term sovereign debt, however, are not fully understood. This is especially true with regard to the potential for equilibrium multiplicity.

For instance, when debt is short-term, Cole and Kehoe (1996) completely characterize a sovereign's susceptibility to rollover crises; Calvo (1988) finds simple conditions under which multiplicity exists when the sovereign cannot commit to debt issuance, but only to its primary deficit; Auclert and Rognlie (2014) find assumptions that guarantee uniqueness in a large class of models, while Passadore and Xandri (2015) provide sufficient conditions for multiplicity in this same class; and Schmitt-Grohé and Uribe (2016) show that small open economies with collateral constraints can suffer from a pecuniary externality that generates multiplicity and underborrowing. With long-term sovereign debt, however, much less has thus far been said about the potential for multiplicity or lack thereof. But this is an important issue to understand, since it governs the potential for expectations to influence market dynamics.

This paper seeks to address this issue. In particular, I demonstrate that markets for long-term sovereign debt can be subject to a multiplicity of equilibria for a given fiscal rule.[1] The multiplicity is generated along what can

[1] A fiscal rule dictates the level of the primary surplus as a function only of states in the current period i.e. a policy function. This paper will not consider models of explicitly maximizing sovereign entities, though the intuition is robust to the inclusion of this feature. See Stangebye (2015) or Aguiar and Amador (2016) for numerical examples of multiplicity in which the sovereign can respond to shifts in expectations by altering his policy rule. Fiscal rules instead interpreted as a constraint on a sovereign maximization problem have been explored by Hatchondo et al. (2015).

be called the 'Lifetime-Laffer Curve,' which dictates the stationary auction revenue generated by a given stationary debt level, factoring in prices that accurately reflect default risk. Much like the contemporaneous 'debt-Laffer curve' by Calvo (1988), which plots debt issuance against the revenue generated at a given auction for sovereign debt, the Lifetime-Laffer Curve will tend to feature multiple stationary debt levels consistent with stationary revenue needs and for the same reasons: There will be one with low debt levels and high prices and another with the reverse, since increasing debt will work to reduce prices.

Expectations regarding toward which of these solutions the economy will converge in the long-run will impact the price schedule the sovereign receives in the initial period: If lenders anticipate that eventually debt levels will converge to the high-debt solution, they will offer a lower price as compensation for the greater associated long-run default risk i.e. dilution. With a fixed fiscal rule, the sovereign is forced to borrow more today to fill his fiscal deficit, which places him on the trajectory toward the high-debt solution, fulfilling lenders' expectations. On the other hand, if lenders anticipate debt levels to converge to the low-debt solution, they will offer a higher price since there is less dilution. The sovereign can then fill his fiscal deficit with less debt, placing him on the trajectory toward the low-debt solution and fulfilling lenders' expectations. Thus, equilibrium multiplicity can arise from a multiplicity of steady states.

It is important to note that this multiplicity is not static in the sense of Cole and Kehoe (1996) or Bocola and Dovis (2015), which describes a problem of liquidity intrinsic to shorter maturity debt. Rather, it is a multiplicity of Markov Perfect Equilibria, each with its own distinct pricing and borrowing schedule. In contrast to liquidity-driven coordination failures, one cannot avoid the problem by lengthening the maturity of the debt. In fact, the long maturity itself is the very source of the multiplicity.

I conduct the analysis assuming that a fiscal rule governs deficit behavior. Fiscal rules of this kind are a benchmark in many classic dynamic equilibrium models such as Leeper (1991), Woodford (2001), or Schmitt-Grohé and Uribe (2007). Some recent work has employed such rules in models of sovereign default (Bocola [2014] or Bi and Traum [2012]). The main advantage of this approach is that a fiscal rule does not allow the sovereign to change his behavior in response to long-run expectations, which allows for an analytic characterization of many of the key results. The intuition holds more generally if one allows the sovereign to respond in this way e.g. Stangebye (2015), but we lose the ability to cleanly piece apart the equilibrium dynamics.

Following the construction of the Lifetime-Laffer Curve, I characterize the implied equilibria. In particular, I show that the only reason why the set of equilibria would be empty would be if the deficits implied by the fiscal rule were too large relative to the sovereign's capacity to generate revenue at a debt auction. When this does not happen, I show that the equilibrium set is ranked i.e. distinct equilibrium price and borrowing schedules never cross. These results rely on a new equilibrium restriction I impose, which is that the sovereign can commit not only to contemporaneous revenue needs, but to contemporaneous debt issuance. This is stronger than, for instance, Calvo (1988) or Lorenzoni and Werning (2014), who assume only the former.

The equilibrium ranking has interesting policy consequences. First, it implies that severe and temporary austerity measures can admit a benefit of short-run debt reduction in the high-debt equilibrium. The reason is because the high-debt equilibrium features lower debt prices for any initial debt level. This implies that if the sovereign is temporarily buying back debt via large austerity measures, it could buy back more debt than it could in the low-debt solution since expectations that it will *eventually* travel back to a region of high debt levels will imply low prices.

Second, it implies that central bank programs initially designed to provide liquidity, such as the Outright Monetary Transactions (OMT) Program, can have the arguably unintended consequence of preventing certain types of solvency defaults. Roch and Uhlig (2014) demonstrate that these sort of programs can coordinate beliefs in the case of liquidity crises while receiving an actuarially fair return. The logic here is similar: The central bank, as a large potential buyer with the capacity to choose the price at which it would purchase debt, can rule out the unpleasant, high-debt equilibrium without actually having to purchase debt: It can offer a demand schedule that is better than the high-debt equilibrium, but worse than its low-debt counterpart, which will force agents to coordinate on the latter.[2] The fact that the equilibria are ranked imply that the central bank would always *want* to intervene in this way. Also, contrary to Lorenzoni and Werning (2014), it will always be *able* to since there is no 'point of no return' i.e. a debt level beyond which equilibrium multiplicity vanishes.

Finally, I show that in the presence of equilibrium multiplicity not all equilibria are equally stable in function space: Low-debt equilibria are stable under iterative algorithms while high-debt equilibria are not. Thus, a new solution technique is required to compute high-debt equilibria, which I develop to assess the potential quantitative impact of the multiplicity.

I apply the model quantitatively to the Eurozone Crisis. Lifetime-Laffer Curve multiplicity resembles the recent crisis in the Peripheral Eurozone since it naturally generates two of its puzzling features. First, malignant market sentiments seemed to play a strong role, suggesting a coordination failure. This is suggested in empirical work by De Grauwe and Ji (2013) or Aizenman

[2]Unlike Roch and Uhlig (2014) in which this sort of liquidity provision can induce greater borrowing and more fundamental default, appropriately implemented liquidity provision here will always reduce debt levels and default frequencies. This is because here liquidity provision is not actually preventing an instance of illiquidity, but rather coordinating beliefs regarding future fundamentals.

et al. (2013) and in more structural work by Corsetti and Dedola (2013) or Lorenzoni and Werning (2014).

Second, most peripheral governments borrowed heavily throughout the crisis in the face of high spreads (Corsetti and Dedola [2013] or Conesa and Kehoe [2014]). This seems to contradict many models of both fundamental and sentiment-driven sovereign default, which generate the sharp prediction that sovereign borrowers ought to delever in the face of unexpected high spreads, not lever up (Arellano [2008] or Aguiar and Amador [2013]). However, it is completely consistent with Lifetime-Laffer Curve multiplicity, in which high spreads are both a cause and consequence of high borrowing.

The model provides a useful benchmark for counterfactual analysis of this event. I perform one such experiment using Portuguese fiscal data during the crisis. Under the assumption that the country coordinated on a high-debt solution, the model suggests that malignant expectations explain as much as 14.98 percentage points (91.12%) of the peak annual yield in early 2012.

Quantitative implementation of the model also suggests that high-debt crises can suffer from indeterminacy. Consequently there is even more room for sentiments to impact spreads despite the limiting behavior being the same: If lenders expect rapid convergence to the high-debt solution, initial prices will be low, which induces rapid convergence; if, however, lenders expect slow convergence to the high-debt solution, initial prices will be higher, which slows convergence. This indeterminacy provides the basis for further extrinsic uncertainty to affect real variables during high-debt crises: Shocks to expectations regarding the *speed of convergence* to the high-debt stationary point can affect outcomes. This is a novel feature of this model, and one that accords with the weak correlation between fundamentals and spreads documented by De Grauwe and Ji (2013).

These dynamics do not exist in the low-debt solution, which implies a determinate equilibrium. Combining this with the prior result on numerical stability yields the interesting result that there is a whole continuum of high-debt equilibria, but *none* of them are stable under iterative algorithms.

Since the crisis in the peripheral Eurozone, several noteworthy pieces have emerged that have tried to explain the unusual phenomenon of borrowing into high spreads and its concomitant upward pressure on debt-to-GDP ratios. Conesa and Kehoe (2012) have termed this phenomenon 'gambling for redemption,' and have argued that being mired in a deep but temporary recession is a necessary condition for such behavior; Broner et al. (2014) and Corsetti and Dedola (2013) have also built models featuring borrowing into high spreads, with the former emphasizing the crowding out effect of sovereign debt issuance when there is domestic preference for debt and the latter arguing that the access to liquidity that the central bank provides is more important for preventing such crises than the printing press; and Rey (2013) argues that debtor nations footing the bill for bank bailouts deepened imbalances.

To the author's knowledge, the only others that have highlighted the role of dilution-driven coordination failures in long-term debt markets in the recent Eurozone crisis are Lorenzoni and Werning (2014). These authors also argue for a Laffer-curve type multiplicity in the spirit of Calvo (1988) with the explicit inclusion of long-term debt. In their environment, as in this paper, a dynamic lender coordination failure can place the economy on a malignant trajectory of high spreads and debt ratios. They term such events 'slow-moving crises.' A key difference between their paper and this is that my equilibrium concept assumes that the sovereign can commit to current debt issuance, whereas in their benchmark model the sovereign can commit only to its current primary deficit. This assumption is crucial to many of the key novel results, such as the existence result, the ranking of the equilibria, and the absence of the 'point of no return' i.e. a debt level beyond which the equilibrium is unique. This last distinction is particularly important since it has policy relevance: It implies that the central bank can still come to the aid of a highly distressed and deeply indebted economy.

The rest of the paper is divided as follows: Section 2 describes the model and highlights the key theoretical and policy results; Section 3 provides a numerical application to the Eurozone and discusses quantitative findings; and Section 4 concludes.

2 Model

I consider a model with two classes of agents. A unit mass of risk-neutral, deep-pocketed lenders and a single sovereign borrower with limited commitment to repay. The sovereign issues a zero-coupon, long-term bond[3] with stochastic maturity, $\lambda \in (0, 1)$. The sovereign cannot commit in period t to repay in future periods.

I will parameterize this limited commitment with a time-invariant risk of default in every period governed by the function $g(b_t)$. For simplicity of exposition, this default shock is the only source of risk in the model, though the key results do generalize to a potentially much larger state space. It is assumed that $g(\cdot)$ is continuous, differentiable, increasing and bounded between zero and one; further, there exists some debt level, $b_u < \infty$ such that $g(b_u) = 1$ i.e. there is a debt level at which the sovereign defaults with probability one.[4]

The sovereign also follows a bounded and time-invariant rule for the primary surplus, $s(b_t)$, much like Leeper (1991) or Schmitt-Grohé and Uribe (2007). The sovereign issues debt, b_{t+1}, in period t at some endogenous price, q_t. Fixing his primary surplus or deficit, the sovereign gets to choose the price q_t

[3]In the Online Appendix, I provide all of the key steady state results for the more commonly used specifications of Hatchondo and Martinez (2009) and Chatterjee and Eyigungor (2012).

[4]This assumption is not strictly necessary but is convenient. It ensures that the contemporaneous debt Laffer curve admits an inverted-U shape and thus there are two debt solutions for a given revenue level. Variations on the results go through provided $\lim_{b\to\infty} g(b) = 1$.

if multiple such prices are available to him. This capacity will be formalized in the equilibrium concept. For the purpose of expositing the dynamical system at the core of the model, we will simply take q_t to be a jump variable.

Even though the government is not explicitly solving a maximization problem, this framework will bear resemblance to such an environment, since limited commitment implies that even when the sovereign can make decisions in period t, he takes the default and borrowing policy functions as given in future periods (Hatchondo and Martinez [2009] or Chatterjee and Eyigungor [2012]). The only additional restriction that would be imposed is that these exogenous rules be ex-post optimal for some set of preferences.

In this environment, the sovereign's budget constraint looks as follows:

$$s(b_t) + q_t[b_{t+1} - (1-\lambda)b_t] = \lambda b_t \tag{1}$$

On the other side of the market, the lenders have access to a risk-free asset yielding a return, r, against which they price sovereign default risk. The return on an investment in risky sovereign debt is dependent both on the repayment of that debt and its future price in secondary markets, q_{t+1}. The no-arbitrage condition places the following restriction on the set of potential equilibrium prices:

$$q_t = \frac{[1 - g(b_{t+1})][\lambda + (1-\lambda)q_{t+1}]}{1+r} \tag{2}$$

The model exposited is quite simple with only one source of risk, but the key results would generalize to a much broader class. For instance, one could easily construct a variation of the real business cycle model developed by Kydland and Prescott (1982) modified to include long-term sovereign debt and a fiscal rule. One could alternatively construct a version of the New Keynesian model of Rotemberg and Woodford (1997) with the same modifications in which government bonds issued to foreigners are real rather than nominal.

2.1 Steady State Analysis

I begin with an analysis of the model's steady states and then investigate the potential these create for self-fulfilling dynamics before formally defining our equilibrium notion. The steady states are natural candidates for analysis and it so happens that they lie along an object that can be appropriately called a Lifetime-Laffer Curve.

I construct the Lifetime-Laffer Curve in this environment by restricting attention to stationary points. Evaluating Equation 2 at any steady state implies that

$$\bar{q} = \frac{[1 - g(\bar{b})][\lambda + (1 - \lambda)\bar{q}]}{1 + r}$$

Solving for $\bar{q}$, I can define a new function

$$\bar{q}(\bar{b}) = \frac{\lambda[1 - g(\bar{b})]}{r + g(\bar{b}) + \lambda[1 - g(\bar{b})]}$$

Now, I can write Equation 1 in terms of steady states to arrive at

$$\begin{aligned} s(\bar{b}) + \bar{q}(\bar{b})\lambda\bar{b} &= \lambda\bar{b} \\ \rightarrow LL(\bar{b}) = \frac{1}{\lambda}\frac{s(\bar{b})}{\bar{b}} + \bar{q}(\bar{b}) &= 1 \end{aligned} \tag{3}$$

$LL(\bar{b})$ sketches out the **Lifetime-Laffer Curve**, which tells us what stationary levels of debt can be sustained. In particular, any $\bar{b}$ such that $LL(\bar{b}) = 1$ may constitute a steady-state solution to the system. Much like the neoclassical growth model, these steady states can serve as limiting points of the dynamical system. But under which conditions will multiplicity actually arise? The following assumptions on the fiscal rule provide one intuitive set of sufficient, though not necessary, conditions.

Assumption 1. *$s(\bar{b})/\bar{b}$ is a continuous and differentiable function for all $\bar{b} \geq 0$.*

This first assumption is necessary to ensure smooth behavior of the Lifetime-Laffer Curve. I make two more assumptions on $s(\cdot)$ to foster an inverted-U shape:

Assumption 2. $\lim_{\bar{b}\to 0} s(\bar{b})/\bar{b} < \frac{r}{r+\lambda}$.

Assumption 3. $s(b_u)/b_u < \lambda$.

The second and third assumptions place upper bounds on the average response of the surplus to debt levels. Together, they will imply that in any steady state, some debt issuance will be required. Even at high debt levels, the sovereign will not respond with surpluses so strongly as to obviate the need to issue new debt to roll over existing debt.

With these assumptions in place, the intuition behind the inverted-U shape is the same as Calvo's (1988). As stationary debt levels increase, stationary prices fall. At some point, the latter effect dominates and the Lifetime-Laffer Curve begins to decrease until it once again meets stationary revenue needs. This is formalized in the following result:

Proposition 1. *If $\exists \hat{b} \in (0, b_u)$ such that $\frac{1}{\lambda}\frac{s(\hat{b})}{\hat{b}} + \bar{q}(\hat{b}) > 1$, then there exist at least two steady state solutions to the dynamical system.*

Proof. The existence of $\hat{b}$ is merely a strong feasibility condition. If no such $\hat{b}$ existed in which the inequality held weakly, then the dynamical system would have no steady states. Under the feasibility condition, the following must hold.

1. Since $\lim_{\bar{b}\to 0} s(\bar{b})/\bar{b} < \frac{r}{r+\lambda}$, then $\lim_{\bar{b}\to 0} LL(\bar{b}) < 1$. Since $LL(\cdot)$ is continuous, by the intermediate value theorem, there exists some $b_L \in (0, \hat{b})$ such that $\frac{1}{\lambda}\frac{s(b_L)}{b_L} + \bar{q}(b_L) = 1$.

2. Since $\frac{s(b_u)}{b_u \lambda} < 1$ and $q(b_u) = 0$, then $LL(b_u) < 1$, and consequently by the intermediate value theorem there exists some $b_H \in (\hat{b}, b_u)$ such that $\frac{1}{\lambda}\frac{s(b_H)}{b_H} + \bar{q}(b_H) = 1$.

□

This multiplicity is exactly derived from the Lifetime-Laffer Curve, with the feasibility condition requiring that a solution be strictly below its 'peak.' Given the smoothness restrictions on the system, another way of saying this is that the set of models that exhibit only one steady state has measure zero, so the condition is not at all restrictive.

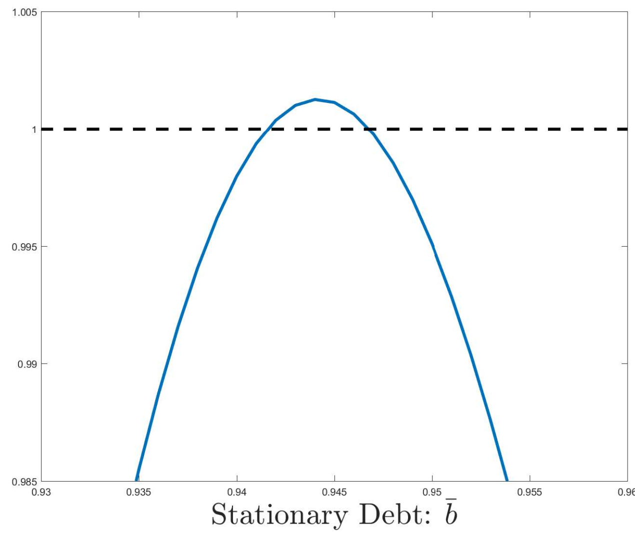
It is not difficult to understand a concrete example in which Proposition 1 is satisfied: If the surplus rule is linear, then the Lifetime-Laffer Curve will exhibit multiplicity as the default rule becomes sufficiently convex since the negative impact on the price in response to high debt levels will be greater than the associated rise in the surplus. This is a very plausible scenario, since nearly any model of endogenous sovereign default generates a very convex default function, in which the sovereign repays for all debt levels from zero to some limit, after which default probabilities rise sharply to one.

One can see this result graphically in Figure 1, in which there are clearly exactly two solutions: A low-debt solution and a high-debt one.[5] In the low debt solution, the sovereign's budget constraint is satisfied since he gets a high steady state price and thus can issue a low amount of debt; in the high debt solution, the opposite is true i.e. the steady state price is low and thus the

[5]Figure 1 will come from the calibrated example in Section 3, which will feature a linear surplus rule and a convex default rule.

sovereign must issue more debt.

Figure 1:
Multiplicity via the Lifetime-Laffer Curve



In order to determine whether these steady states can be interpreted as limiting points of the dynamical system, one would need to know their local stability properties.

Proposition 2. *If $\lambda \in (0,1)$, then the characteristic roots of a candidate solution to the Lifetime-Laffer Curve, $(\bar{q}, \bar{b})$, are given by:*

$$\hat{\lambda}_1 = \frac{B + \sqrt{B^2 - 4D}}{2}$$
$$\hat{\lambda}_2 = \frac{B - \sqrt{B^2 - 4D}}{2}$$

where

$$B = (1-\lambda) + \frac{\lambda - s'(\bar{b})}{\bar{q}} + \frac{1+r}{[1-\lambda][1-g(\bar{b})]} - \frac{(1+r)g'(\bar{b})\lambda\bar{b}}{[1-\lambda][1-g(\bar{b})]^2}$$
$$D = \left((1-\lambda) + \frac{\lambda - s'(\bar{b})}{\bar{q}}\right) \times \left(\frac{1+r}{[1-\lambda][1-g(\bar{b})]}\right)$$

Proof. See Appendix A □

With this result, one can immediately apply Theorem 6.6 of Stokey et al. (1989) to determine the local stability of the underlying solution. If both roots are less than unity in magnitude, then any point in the neighborhood of the candidate solution will eventually converge to the candidate solution; if only one is less than unity in magnitude, then there will be a one-dimensional stable manifold i.e. a saddle-path; if both are greater than unity, the solution will be unstable. Examples can easily be constructed in which both solutions are stable. This will be seen in Section 3.

A full solution to the model, requires more than just the steady state and its stability properties. An equilibrium concept is needed that will include how the jump variable, q_t is chosen for a given state, b_t. The next section will describe such a concept.

2.2 Equilibrium Definition and Global Analysis

The analysis thus far has focused on local behavior around steady states that can sensibly be interpreted as limiting points. I now endeavor to formulate a global equilibrium concept from the fundamental equations governing the model. In doing so, I will restrict attention to equilibria in which the sovereign can be interpreted as having monopoly power over debt issuance. This is a standard assumption in the quantitative literature in the vein of Eaton and

Gersovitz (1981), but has yet to be applied to a framework with fiscal rules.

I formalize this with an equilibrium definition:

Definition 1. *A* ***Markov Perfect Equilibrium (MPE)*** *is a pair of time-invariant functions, $q(b_{t+1})$ and $a(b_t)$, such that*

1. $b_{t+1} = a(b_t)$ *and implies* $b_{t+1} \in [0, b_u)$ *for any* $b_t \in [0, b_u)$
2. $q(b_{t+1}) = \frac{[1-g(b_{t+1})]\times[\lambda+(1-\lambda)\times q(a(b_{t+1}))]}{1+r}$
3. $a(b_t) = \inf \ \{b_{t+1} | \ q(b_{t+1})[b_{t+1} - (1-\lambda)b_t] = \lambda b_t - s(b_t)\}$

Notice that the sovereign's price-making ability is guaranteed in this definition by point (3): If there are multiple price-debt pairs at which the sovereign can satisfy his budget constraint, which there will often be,[6] he is assumed to be able to choose the best among these in the sense of the one that yields the lowest level of debt and thus the highest price. I will call this third condition **Commitment to Contemporaneous Debt Issuance**, since it implies that whenever a contemporaneous Laffer curve emerges, the sovereign can always commit to being on its good, left-hand side.

Notice also that a Markov Perfect Equilibrium provides a prescription for the jump variable (q_t) in terms of the state variable (b_t) for the dynamical system described by Equations 1 and 2: $q_t = q(a(b_t))$. Thus, one interpretation of a MPE is a solution to the dynamical system, though there are potentially many other types of solutions.

The first result is relatively trivial, but it will assist in the establishment of later results.

Lemma 1. *If $\lambda b_t - s(b_t)$ is an increasing function of b_t, then in any Markov Perfect Equilibrium it must be that $a(b_t)$ is an increasing function. This result*

[6]See Calvo (1988).

requires only that Equilibrium Condition 3 is satisfied.

Proof. Consider two points, $b_t \leq b'_t$. Begin by noting that for any fixed b_{t+1}, the following inequality holds

$$q(b_{t+1})[b_{t+1} - (1-\lambda)b'_t] - q(b_{t+1})[b_{t+1} - (1-\lambda)b_t] \leq 0$$

i.e. The more initial debt the sovereign has, the less he will raise at the auction. In equilibrium though, this difference must be weakly positive to satisfy the budget constraint, since $\lambda b_t - s(b_t) \leq \lambda b'_t - s(b'_t)$. Under Equilibrium Condition 3, auction revenue is increasing in b_{t+1}, since by assumption the sovereign is on the upsloping portion of the contemporaneous Laffer curve. Therefore, in order for the budget constraints to be satisfied, it must be that $b'_{t+1} \geq b_{t+1}$, and thus $a(b'_t) \geq a(b_t)$. $\square$

I will maintain the assumption of Lemma 1 throughout the rest of the paper for simplicity, since it is not very restrictive. I will also maintain one more assumption throughout the analysis:

Assumption 4. $s(b_t) < \lambda b_t$ *for any* $b_t \in [0, b_u)$.

Under this assumption, the sovereign never engages in active debt buybacks. He may delever, but it will be by letting existing debt mature and not rolling it over rather than actively repurchasing debt. This is not a heavy assumption; Bulow and Rogoff (1988) argue that debt buybacks would never be optimal in a similar environment.

Before proceeding, I show first that any multiplicity of Markov Perfect Equilibria is contingent on the long-term nature of the debt.

Proposition 3. *If $\lambda = 1$, then at most one Markov Perfect Equilibrium exists.*

Proof. To see this, first note that the pricing equation implies $q(b_{t+1}) = \frac{1-g(b_{t+1})}{1+r}$. Thus, the pricing function is unique. Further, one can substitute this into the budget condition to arrive at

$$\left[\frac{1-g(b_{t+1})}{1+r}\right] b_{t+1} = b_t - s(b_t)$$

If this expression has no solution for b_{t+1} for some b_t, then no MPE exists. On the other hand, if a solution for b_{t+1} exists for all $b_t \in [0, b_u)$, the solution set will generally not be single-valued. Definitionally, the Markov Perfect Equilibrium will always choose the infinum of these. Since there is necessarily a unique was to do this, if an MPE exists one can write $a(b_t)$ uniquely for any b_t and thus the equilibrium is unique. □

Non-existence in this environment arises when primary deficits are too large relative to the sovereign's capacity to raise revenue at debt auctions. Provided this does not happen, the result tells us that the equilibrium exists and is unique. Short-term debt may allow for multiplicity along the Lifetime-Laffer Curve, but if the solution converges to a stationary point, it must be unique. Thus at most one solution to the Lifetime-Laffer Curve can serve as the limiting case for the dynamical system.

This result implies that a multiplicity of Markov Perfect Equilibria must necessarily arise from the long-term nature of the debt, specifically the debt dilution.

When the debt is not short-term, this does not hold and there will be potential for a multiplicity of equilibria. In particular, multiple Markov Perfect Equilibria often exist converging to different limiting points along the Lifetime-

Laffer Curve.

I begin with an existence result:

Proposition 4. *The set of Markov Perfect Equilibria is only empty if $s(b_t)$ implies a sequence of primary deficits that is too large relative to the sovereign's capacity to raise revenue at the debt auction for some initial debt level $b_t \in [0, b_u]$.*

Proof. See Appendix A $\square$

In this sense, the emptiness result is akin to that for short-term debt. However, here dilution premia also affects the sovereign's capacity to issue above and beyond immediate default risk and may depress prices to the point of infeasibility. Notice that while this result does not guarantee the existence of multiplicity, it is not precluded in the way that it is with short-term debt.

This existence result and the corollaries that follow rely crucially on the assumption that the sovereign can commit to contemporaneous debt issuance. Without it, the result fails to hold. This is because the relevant functional operators cannot be shown to be monotone without this assumption, and that monotonicity is critical for the fixed point theorem that implies the result.

In the event that multiplicity arises, it is possible to say something further about what it looks like.

Corollary 1. *If the set of Markov Perfect Equilibria is non-empty, then it is ordered i.e. if $(a_L(b_t), q_L(b_{t+1}))$ and $(a_H(b_t), q_H(b_{t+1}))$ are two distinct equilibria, then wlog $a_L(b_t) \leq a_H(b_t)$ for all $b_t \in [0, b_u)$ and $q_L(b_{t+1}) \geq q_H(b_{t+1})$ for all $b_{t+1} \in [0, b_u)$.*

Proof. See Appendix A □

The ranking of equilibria is noteworthy: The sovereign *always* gets a better price in a low-debt equilibrium than in any high-debt counterpart i.e. the price schedules never cross. In turn, he always borrows more in the high-debt equilibrium. One can apply this directly to the case in which multiplicity arises along the Lifetime-Laffer Curve.

Corollary 2. *Let $\bar{b}_L < \bar{b}_H$ denote two solutions along the Lifetime-Laffer Curve. If there are two Markov Perfect Equilibria, $(a_L(b_t), q_L(b_{t+1}))$ and $(a_H(b_t), q_H(b_{t+1}))$, such that for any $b_t \in (0, b_u]$ we have $\lim_{n\to\infty} a_L^n(b_t) = \bar{b}_L$ and $\lim_{n\to\infty} a_H^n(b_t) = \bar{b}_H$, then $a_L(b_t) \leq a_H(b_t)$ for any $b_t \in [0, b_u)$ and $q_L(b_{t+1}) \geq q_H(b_{t+1})$ for any $b_{t+1} \in (0, b_u]$.*

Thus, equilibrium multiplicity implied by the Lifetime-Laffer Curve will always be *uniformly ordered* when it exists. This result has a couple of significant policy implications

2.2.1 Policy Implication 1: Liquidity Provision

Since Lifetime-Laffer Curve multiplicity is fundamentally an issue of coordination and beliefs, liquidity provision from the central bank can in fact eliminate the high-debt trajectory. Since the price schedules are ordered, it will always want to do so, regardless of what point the economy is in the state space.

Here, one can think of the central bank as a deep-pocketed actor with the capacity to purchase sovereign debt at a price of its choice in any period. I also assume that if the sovereign faces multiple demand schedules, he will always purchase from the one that provides the highest price. Under this assumption, the following holds.

Proposition 5. *The central bank can costlessly immediately eliminate the high-debt trajectory at any level of debt.*

Proof. The central bank can offer a demand schedule

$$q_t^{CB}(b_{t+1}) = q_t^L(b_{t+1})$$

If it does this, the sovereign would always choose to borrow from the central bank rather than the high-debt equilibrium, since it gets a better price. Thus, the high debt equilibrium cannot be sustained. If it were to borrow from the central bank at this rate, it would imply a debt policy $a^L(b_{t+1})$ by definition. But this is the low-debt equilibrium, and can thus be financed by private foreign investors. Thus, the economy can coordinate on the low-debt equilibrium and the central bank pays no cost. □

Thus, the provision of liquidity as in OMT can prove effective in the case of a crisis of this nature, *even though all defaults are fundamental and not driven by liquidity concerns*. Notice that OMT here is not interpreted as a bailout or a rescue package. Even in the low-debt trajectory, there is accurately priced default and dilution risk, and the central bank is assumed to not come to the rescue. This is in many senses true to the way that OMT is actually set up, since stringent conditions need to be met for a member state to apply for OMT, and these conditions are least likely to be met when the sovereign is experiencing a crisis (Wolf [2014]). For instance, in recent years turbulent Greece has not met the conditions to apply for OMT help since they have not managed to regain complete access to private lending markets, which is a necessary condition of the OMT program.

Thus, the model predicts that providing liquidity in this way may not be an effective tool for preventing all future defaults (at least not costlessly), but it *is* an effective way to prevent a malignant, sentiment-driven build-up of debt and spreads. Further, and in contrast to Lorenzoni and Werning (2014), the central bank is always *able* to do this. This is because equilibrium multiplicity pervades for all initial debt levels, whereas in their model there is a 'point of no return' i.e. a debt level beyond which the equilibrium is unique and liquidity provision is thus ineffective.

2.2.2 Policy Implication 2: Austerity

The ordering result also has implications for the impact of austerity policies. Let us now consider a different environment, in which in period t the sovereign runs some primary surplus $s_t > \lambda b_t > 0$; then from period $t+1$ on, the economy follows the surplus rule $s(b_t)$ and coordinates on one equilibrium or another. Since the sovereign cannot commit to any actions besides contemporaneous debt issuance in period t, he cannot influence long-run beliefs regarding his actions. These beliefs will influence the price schedule in period t, which the sovereign takes as given.

Let $b_{L,t+1}$ denote debt choice when beliefs coordinate on the low-debt equilibrium from period $t+1$ onward, and let $b_{H,t+1}$ denote debt choice when beliefs coordinate instead on the high-debt equilibrium. In this environment, the following holds:

Proposition 6. *A fiscal surplus buys back more debt in a high-debt equilibrium than in a low-debt equilibrium i.e.* $b_{L,t+1} \geq b_{H,t+1}$.

Proof. We know that

$$
\begin{aligned}
q_L(b_{L,t+1})[b_{L,t+1} - (1-\lambda)b_t] &= \lambda b_t - s_t < 0 \\
q_H(b_{L,t+1})[b_{L,t+1} - (1-\lambda)b_t] &\geq \lambda b_t - s_t \\
&\rightarrow b_{L,t+1} \geq b_{H,t+1}
\end{aligned}
$$

where the last line follows from the fact that auction revenue is always increasing in the region of buybacks. □

This result tells us that austerity, while painful, can be particularly effective during a crisis *if* the government engages in it severely enough to turn a primary surplus into a fiscal surplus and induce debt buybacks *and if* the goal is temporary debt reduction. The intuition is simple: Debt prices are always worse in the high-debt equilibrium, and thus it is cheaper to buy back debt in this equilibrium. The fact that lenders expect the sovereign to *eventually* traverse into a high debt region could in fact help *reduce* indebtedness in the short-run.[7]

3 Quantitative Analysis

I now show that the multiplicity implied by the Lifetime-Laffer Curve translates to a multiplicity of Markov Perfect Equilibria. I begin with a brief description of the solution method and then provide numerical examples.

3.1 Solution Method

The principle difficulty one encounters when attempting to solve this model is guaranteeing commitment to contemporaneous debt issuance. This is because

[7]It should be noted that no large, distressed Eurozone country actually ran a fiscal surplus during the crisis, despite large austerity measures. Many were able to run substantial primary surpluses, but these did not induce debt buybacks.

in order to construct the contemporaneous Laffer curve in the first place, and thus restrict attention to its Pareto-superior left side throughout the model solution, one needs to know the demand curve for debt, $q(b_{t+1})$. This object is contingent upon future prices and borrowing behavior. But these in turn are dependent upon their respective demand curves for debt, which is yet unknown. And if one attempts to iteratively search for one from a random initial guess, say the zero price, there's no guarantee that such a method will be able to uncover a *multiplicity* of equilibria. In fact, it can be shown that in the event of multiplicity, only the low-debt equilibrium is numerically stable.

Proposition 7. *Policy function iteration from below robustly converges to the lowest-debt equilibrium. Generically, the same procedure does not converge to the highest-debt equilibrium from above or below.*

Proof. See Appendix A □

Generically, the highest-debt equilibrium is numerically *unstable* in the sense that iterative algorithms from above or below will never find it. It is important to note that the high-debt equilibrium is not unstable in the sense that equilibrium refinements will kill it, as the high-debt equilibrium is in Calvo (1988); the high-debt equilibrium in this model survives any refinement that the low-debt equilibrium would. Rather, the high-debt equilibrium is unstable *in function space*. This is because the functional operator defined by a policy function iteration exhibits contractionary properties around the low-debt equilibrium, but expansionary properties around the high-debt equilibrium. An analogous result holds for price function iteration.

To get around this stability problem and uncover the multiplicity, I apply a novel technique. It begins by noting that the previous stability result only suggests that iterative algorithms will not uncover the high-debt equilibrium when the starting point is uniformly above or below it. It does not apply given a starting point that crosses it, especially if that starting point is relatively close to the equilibrium.

I exploit this by attempting to find a good initial guess for an iterative algorithm i.e. one that intersects it (or at least touches it) and is nearby. To find such an initial guess, our approach will be to use a shooting algorithm on an approximating dynamical system. Starting from any b_0, I can apply the shooting algorithm over q_0 and construct trajectories toward one of the limiting points implied by the Lifetime-Laffer Curve. The solution to the approximating model provides a demand curve for debt that can be used as an initial guess in an iterative algorithm to solve the general model, since it will be nearby and will necessarily intersect with it at least at the limiting point. Details regarding the solution technique can be found in Appendix B.

3.2 Numerical Example

I first consider a stationary environment of the sort that is common in modern macroeconomics. I then augment the model to include a temporary non-stationary component and conduct a counterfactual analysis of the recent crisis in the Peripheral Eurozone.

The example provided is meant to be illustrative rather than an estimation, intended primarily to demonstrate how one could apply the analytic toolbox developed in Section 2. Thus, for simplicity let us suppose that the surplus function takes the parametric form $s(b) = \kappa_0 + \kappa_1 b$. This is the popular form employed by Leeper (1991), Schmitt-Grohé and Uribe (2007), and Bocola (2014) among others. Also for simplicity, I assume a default probability given by $g(b) = b^\alpha$ for all $b \in [0, b_u]$, where $b_u = 1$.

I assume that a period in the model is a quarter and that $r = .02$ and

$\lambda = 0.0385$. This implies an average maturity of 6.5 years. This is the maturity chosen by Chatterjee and Eyigungor (2012) in their seminal analysis of Argentina.

The parameters are chosen as follows: $\kappa_1 = 0.35$, which reasonably assumes that a 1% increase in debt is met by a .35% increase in the primary surplus.[8] The default curvature parameter, α, is chosen to be the smallest multiple of ten possible that induces multiplicity along the Lifetime-Laffer Curve. In this case, $\alpha = 70$. Such severe convexity in the default function is typical in such models of endogenous default as Arellano (2008) or Aguiar and Gopinath (2006), in which default probabilities are near zero for almost the entire debt domain and spike near the endogenous credit limit.

The last parameter is the surplus constant, κ_0. This is chosen to ensure that the high-debt equilibrium delivers a 10% spread, which is in the neighborhood of crisis-level spreads in many emerging markets (Aguiar et al. [2015]). I will not attempt to match debt levels to data from any country since there are no more free parameters. Nevertheless, the model will yield predictions for the variation in debt levels across the two equilibria in percentage terms.

In this example, I derive the Lifetime-Laffer Curve found in Figure 1. Here, there are exactly two solutions: $b_L = 0.9416$ and $b_H = 0.9468$. This implies that debt levels are only 0.55% higher in the high-debt equilibrium. The concomitant steady-state prices exhibit larger differences, with $q_L = 0.5217$ and $q_H = 0.4742$. The former implies a stationary annual spread of 6.6% while the latter implies 10.0%. Recall that the latter was calibrated, so the former can be considered a counterfactual.

The example thus suggests that for reasonable parameters, the impact of equilibrium selection on debt levels is almost negligible but the impact on spreads is enormous, being nearly half. This is because the debt is of relatively

[8] The choice of $\kappa_1 = .35$ is not important. A different value is chosen in the subsequent section.

long maturity, which implies that little of it needs to be rolled over at any given auction. This mitigates the impact of spreads on debt build-up.

With this in mind, the system can now be characterized. The first result is the key multiplicity finding.

Result 1. *There are a continuum of Markov Perfect Equilibria. The least of them converges to the low-debt steady state; all others converge to the high-debt steady state.*

The characteristic roots of the two steady states are given below:

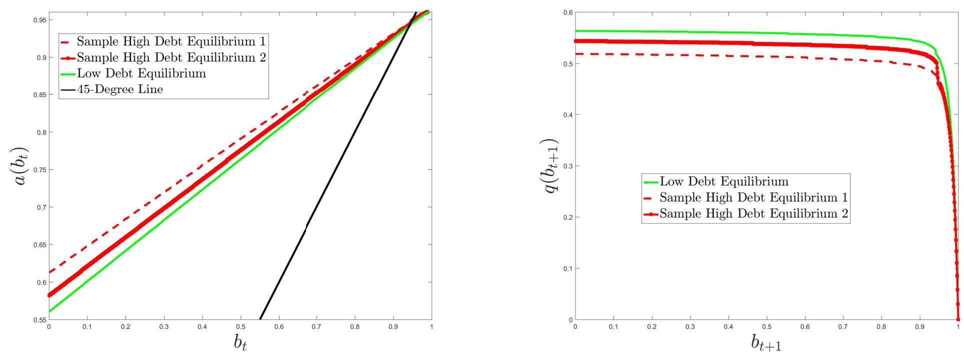
$$\hat{\lambda}_{L,1} = 1.0086$$
$$\hat{\lambda}_{L,2} = 0.3890$$
$$\hat{\lambda}_{H,1} = 0.9906$$
$$\hat{\lambda}_{H,2} = 0.3335$$

From this, it is clear that the high-debt solution is stable in the sense that any nearby point will eventually converge to (q_H, b_H). The low-debt solution, however, only has a one-dimensional stable manifold i.e. a saddle path.

The local stability properties adhere globally as well. Solving the model quantitatively using the algorithm outlined in Appendix B reveals that both steady states can act as limiting points, and that many Markov Perfect Equilibria exist which converge to the high-debt steady state, while only one converges to its low-debt counterpart. This is particularly interesting since the unique low-debt equilibrium is the only one that is numerically stable under policy function iteration; even though there is a whole continuum of high-debt Markov Perfect Equilibria, solving for any of them requires our nonstandard, novel technique.

The policy functions and price schedules associated with some of these

Figure 2:
Markov Perfect Equilibria: Calibrated Example



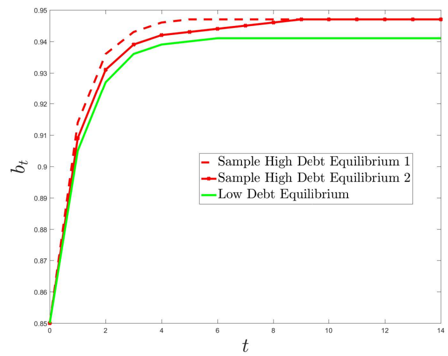
(a) Policy Functions

(b) Price Schedules

equilibria can be found in Figure 2. The low-debt equilibrium converges to a lower debt level in the long run, and thus there is less associated dilution and a greater price than its high-debt counterparts. The reason for the discrepancy *across* high-debt equilibria i.e. the basis for indeterminacy is more subtle and revolves around the *speed of convergence*: Sample High-Debt Equilibrium 1 converges to b_H more quickly than High-Debt Equilibrium 2, which implies greater dilution and a worse price. This can be seen in Figure 3. This worse price causes the need to issue more at low debt levels in High-Debt Equilibrium 1 than in High-Debt Equilibrium 2. But this excessive issuance causes faster convergence to b_H in High-Debt Equilibrium 1, and thus the dynamics are self-fulfilling. These self-fulfilling dynamics are not present in any trajectory converging to b_L; consequently, the low-debt equilibrium is unique.

The indeterminacy associated with the high-debt solution suggests that during crises of this kind sunspot shocks could cause extrinsic fluctuations in yields, much like Farmer and Guo (1994) or Farmer and Benhabib (1994). One could imagine the economy shifting across high-debt Markov Perfect Equilibria

Figure 3:
Markov Perfect Equilibria: Debt Trajectories



without altering long-run expectations, providing a further role for market sentiment during crises.

3.2.1 Comparison with Lorenzoni and Werning (2014)

Many of these results differ from those of Lorenzoni and Werning (2014), who explore a similar model. The assumptions made in this model are more restrictive e.g. I explicitly assume that the sovereign has the ability to commit to debt issuance in any period and not just revenue; but the environment is also different e.g. the *arrival rate* of a default depends on the debt level and not just the probability of default conditional on some exogenous arrival. Since the models are quite different and many different assumptions could be driving the discrepancies, a brief discussion of the key differences is in order.

First, in their model a sort of a Lifetime-Laffer Curve exists, in which there are two steady states. In both models, the low-debt steady state is saddle-path stable. However, in their model the high-debt steady state is always unstable, while in this model, it is *globally stable*. Thus, their high-debt equilibrium does not traverse toward the high-debt steady state, as it does in this model; rather, debt and yields ratchet off toward infinity in a non-stationary way.

This discrepancy is significant, since this model implies that sunspot shocks can further influence equilibrium dynamics during high-debt crises via shocks to expectations regarding the speed of convergence, which is not a feature of their model. This is consistent with weak correlations between fundamentals and spreads documented by De Grauwe and Ji (2013).

The difference in the local stability properties is *not* a result of the assumptions regarding commitment, since local stability can be computed before the global equilibrium is defined with the additional commitment restrictions. Rather, it is a product of disparate model assumptions, particularly their assumption that there is a 'cap' on the primary surplus $\bar{s}$, regardless of the debt level. Thus, as debt levels grow large, eventually the primary surplus stops attempting to reverse course and debt levels begin to explode. That is not the case in our model, in which there is no cap on the primary surplus. In fact, the primary surplus here is always increasing and thus grows *even larger* as debt levels grow. This contractionary force pushes large debt levels back toward the high-debt steady state, lending it stability.

Second, in this model there is no 'point of no return' i.e. a debt level beyond which the equilibrium is unique. In this model, long-run beliefs are influential at *any* debt level, while in their model beliefs can only matter for some intermediate debt ranges. Unlike the stability result, this discrepancy *is* driven by the commitment assumptions. In particular, they assume that the sovereign takes q_t as given and cannot choose the best if multiple prices are available to him at time t. The 'point of no return' arises when there are some intermediate debt levels with two possible equilibrium prices, but higher debt levels imply a unique price.

In my set-up, the sovereign can always choose the highest price when multiple prices are available. Thus, by construction the solution is always unique in their sense of multiplicity: In that intermediate range, the sovereign can and would choose the better price. Thus, the multiplicity is not quite the same across the two models: A multiplicity of equilibria in my environment implies two distinct demand schedules, $q(b_{t+1})$, rather than just two distinct prices. Since the demand for debt does not depend on current debt levels but only future ones, the price does not depend on current debt levels and thus the multiplicity pervades for any initial debt level.

The counterfactual predictions regarding debt levels are another practical difference. In their counterfactual, for instance, both debt levels and spreads would be substantially lower; in this model, only spreads would be substantially lower. One could think of their model's Eurozone crisis counterfactual as the absence of a crisis, in which debt levels hover around pre-crisis levels; whereas this model would predict a counterfactual more like the US, in which debt levels still exploded but yields remained low and spreads near zero. A more detailed counterfactual analysis will follow in the next section.

3.3 Calibrated Eurozone Crisis Counterfactual

The stationary model analyzed in the previous section provides a wealth of insight into how Lifetime-Laffer Curve Multiplicity can arise and what its consequences are. I now seek to apply it to a contemporary setting to see the sort of quantitative bite it has. A natural choice is the Eurozone crisis, since Eurozone debt was relatively long in maturity and many authors have argued that sentiments played a key role (De Grauwe [2011], De Grauwe and Ji [2013], or Lorenzoni and Werning [2014]).

The parameters of the model are roughly chosen to match the experience of Portugal. The exercise could be performed for any of the distressed Eurozone countries at the time since most of them exhibited a prolonged period of borrowing into high spreads. Portugal, however, fits particularly well because in addition to this debt levels stayed relatively high in the years following the crisis i.e. after the supposed expectations shift. This is consistent with numerical predictions that these coordination failures have a much larger impact on spreads than debt levels.

The quantitative example before relied on a calibration to steady state levels, but such a stationary model is ill-suited to perform a counterfactual analysis of the highly non-stationary Eurozone crisis. Here, I provide a suitable superset of the general model that allows for such a comparison. The basic augmentation is the following: Suppose that after 2012 the government follows a calibrated fiscal rule similar to the previous section and coordinates on either the Low-Debt Equilibrium or Sample High-Debt Equilibrium 1, which provide respectively the *highest* and *lowest* possible equilibrium prices for a given debt level. The choice of these equilibria will give a sense of how large the impact of sentiment shifts can be.

The years building up to 2012 are not restricted by the fiscal rule. Instead, I will directly feed the sequence of primary deficits from 2008-2011 from the data as well as the initial debt level at the end of 2007. Lenders can price debt issuance accurately in these periods since the entire future stream of deficits and prices is known i.e. perfect foresight.

I begin with the heavy assumption that not only was Portugal in a perfect foresight equilibrium, but that the OMT announcement was completely unexpected. Further, I assume that OMT resulted in a switching of equilibria from the high-debt MPE discussed in the previous section to its low-debt counterpart.

Portuguese debt levels are scaled such that debt in 2007 is at its highest possible level such that the sequence of deficits from the data can be feasibly financed for either set of expectations. I further interpret b_t throughout as true levels rather than the debt-to-GDP ratio and so all other nominal variables will be scaled by the same factor. This is so that I can ignore for the time being any feedback from debt levels to output growth. It ought to be noted that this assumption will place a *lower bound* on the difference between the calibrated model and the counterfactual, since Reinhart and Rogoff (2010) document a negative relationship between debt burdens and output growth.

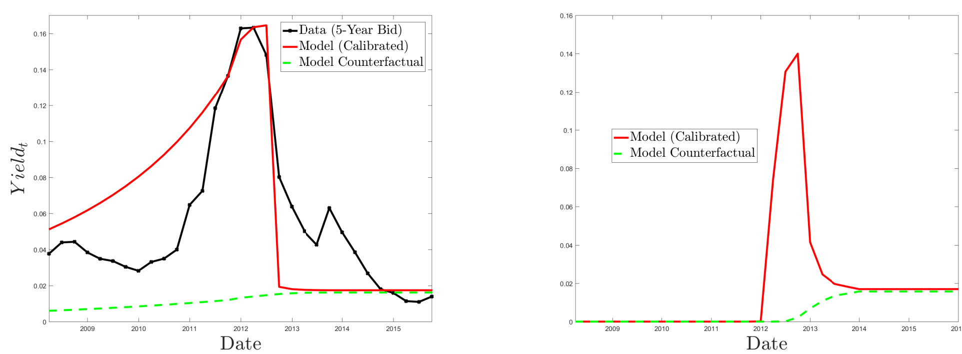
I will not attempt to match the nominal debt trajectory during the crisis explicitly, instead focusing on the spreads. Debt levels were influenced by many other factors that the model does not consider, such as bailout package terms and shifts in the maturity composition. Further, the face value of the debt is not always the best metric of a country's debt burden, since countries with identical face values can have very different coupon schedules and maturities.

From 2012 onward, I will assume that the trajectory is governed by a stationary MPE at a quarterly frequency. The functional form will be the same as the previous section i.e. $s(b) = \kappa_0 + \kappa_1 b$ and $g(b) = b^{\alpha}$. I will choose the parameters in the same vein, but with slightly different targets. I will assume that $\kappa_1 = 0.5$, as Bocola (2014) does for Italy; I will set α to its lowest possible decile such that the Lifetime-Laffer Curve has two solutions; and I will choose κ_0 such that the stationary high-debt annual bond yield is 17%, which is slightly above the peak yield in mid-2012. The idea here is to place a realistic cap on the level of the spreads the model can generate during the crisis, since the model necessarily will be converging to this high yield from below. Lastly, I will set the risk-free rate to be very low in accordance with its levels during the crisis i.e. $r = 0.0025$.

The only difference between the calibrated crisis model and the counterfactual will be divergent expectations circa 2007. The calibrated model will assume that the sovereign is initially on the monotone trajectory to the bad, high-debt side of the Lifetime-Laffer Curve, while the counterfactual will assume from the outset that the sovereign is always on the trajectory converging to its good side. Thus, the counterfactual exercise holds all fundamentals constant including government spending and output until 2012 and asks what spreads and default probabilities would have looked like with different initial beliefs.

The results of this exercise can be seen in Figure 4.[9] Observe first that the model does a relatively good job fitting the data, for its simplicity. The yield rises more slowly than the data, but the peak turns out to be nearly the same in 2012.[10] It is true that the model reacts more quickly to shifts in expectations caused by OMT than the data, but this is to be expected given the simplicity of the model and lack of any search or information frictions.

Figure 4:
Crisis Counterfactual



(a) Annual Yield **(b)** Annual Default Probability

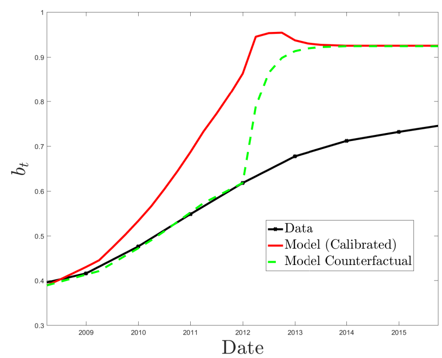
The counterfactual green dotted line is quite interesting in both of the figures. One can see that counterfactual spreads are much smaller; up to an order of magnitude. The peak difference between the two is 14.98 percentage points (91.12%) and occurs just before the expectations shift. Interestingly, the

[9]Yield data from Thomson Reuters Datastream.

[10]Given the indeterminacy of the high-debt solution, I could alternatively match the spread trajectory in the data exactly by finding appropriate sunspot shocks. This would not change the counterfactual equilibrium, which is determinate, and thus would not change significantly change the key quantitative results or any of the policy implications.

counterfactual spreads are steadily rising as well since the government is in fact borrowing heavily in the counterfactual. The reason for the massive difference is not the debt levels, which look similar and can be found in Figure 5, but the implied dilution risk, which is enormously smaller in the counterfactual.

Figure 5:
Scaled Debt Levels: Crisis Counterfactual



It ought to be noted that the model does not capture well the behavior of debt levels, as can be seen in Figure 5. This is for three reasons: First, the model does not account for the discounted rate that came with the bailout package of 2011, which contributed enormously to the debt levels at the time; second, the model does not have multiple maturities, and in reality the sovereign can mitigate gross debt build-up buy shortening maturity during a crisis, as has been noted by Arellano and Ramanarayanan (2012) and Aguiar and Amador (2013); and third, the fiscal rule that is assumed to apply from 2012 onward generates deficits that are counterfactually large in 2012 and 2013 and counterfactually small in 2014 and 2015.

This discrepancy should not be cause for alarm for two reasons. First, the debt path does not change substantially in the counterfactual, and as such the spread path is the real object of interest during the crisis. The primary determinant of the spreads are dilution fears that extend well beyond 2015. Second, while it is outside the model, the bailout package that added the greatest amount to the debt burden was announced in full in 2011. Consequently, debt burdens that were not officially added until sometimes 2013 were known by 2011. The model's front-loaded debt build-up, which spikes around the time of the bailout, can be interpreted as a proxy for this. Thus, the estimated default probabilities, given in Figure 4, may in fact be reasonable and if so suggest that the OMT program and Draghi's famous 'whatever it takes speech,' may have come just in time.

4 Conclusion

In this paper, I demonstrated that long-term sovereign debt can exhibit equilibrium multiplicity along the Lifetime-Laffer Curve. This multiplicity is ordered, which implies that intervention via liquidity provision to coordinate beliefs is always feasible and desirable, *regardless* of the initial debt level. It is also the case that temporary austerity measures in the bad equilibrium can buy back more debt than they can in the good equilibrium.

Applying this framework to the Eurozone, I established that these coordination failures could explain a potentially large fraction of Portuguese spreads during the crisis. I further showed during these crises that indeterminacy can allow even more room for sentiments to influence market outcomes, as expectations regarding the speed of convergence to the bad limiting outcome can be self-fulfilling.

There is still much room for further work. In particular, a method of identifying such sentiment-driven build-ups in the data would prove quite useful in verifying the applicability of the model. An investigation of the potential for this multiplicity in other types of financial markets suffering from similar frictions, such as the market for commercial paper or municipal debt, could also be in order.

Appendices

A Proofs of Theoretical Results

A.1 Proof of Proposition 2

Notice that the Jacobian elements evaluated at the candidate solution are given below:

$$
\begin{aligned}
J_{11}(\bar{x}) =& (1-\lambda) + \frac{\lambda - s'(\bar{b})}{\bar{q}} \\
J_{12}(\bar{x}) =& -\frac{1}{\bar{q}^2}\left[\lambda\bar{b} - s(\bar{b})\right] \\
J_{21}(\bar{x}) =& \frac{\bar{q}(1+r)g'(\bar{b})}{[1-\lambda][1-g(\bar{b})]^2} J_{11}(\bar{x}) \\
J_{22}(\bar{x}) =& \frac{1+r}{[1-\lambda][1-g(\bar{b})]} + \frac{\bar{q}(1+r)g'(\bar{b})}{[1-\lambda][1-g(\bar{b})]^2} J_{12}(\bar{x})
\end{aligned}
$$

Notice that we can write the determinant as follows:

$$
\begin{aligned}
\det(\bar{x}) = J_{11}(\bar{x}) \times & \underbrace{\left(\frac{1+r}{[1-\lambda][1-g(\bar{b})]} + \frac{\bar{q}(1+r)g'(\bar{b})}{[1-\lambda][1-g(\bar{b})]^2} J_{12}(\bar{x})\right)}_{J_{22}(\bar{x})} \\
& - \underbrace{\frac{\bar{q}(1+r)g'(\bar{b})}{[1-\lambda][1-g(\bar{b})]^2} J_{11}(\bar{x})}_{J_{21}(\bar{x})} \times J_{12}(\bar{x}) \\
\rightarrow \det(\bar{x}) = J_{11}(\bar{x}) \times & \left(\frac{1+r}{[1-\lambda][1-g(\bar{b})]}\right)
\end{aligned}
$$

The latter term in this expression is always positive, and thus the determinant has the same sign as $J_{11}(\bar{x})$. By solving out for $\det(J - \hat{\lambda}I) = 0$, we arrive at a quadratic term with solutions:

$$\hat{\lambda}_1 = \frac{J_{11}(\bar{x}) + J_{22}(\bar{x}) + \sqrt{[J_{11}(\bar{x}) + J_{22}(\bar{x})]^2 - 4\det(J(\bar{x}))}}{2}$$
$$\hat{\lambda}_2 = \frac{J_{11}(\bar{x}) + J_{22}(\bar{x}) - \sqrt{[J_{11}(\bar{x}) + J_{22}(\bar{x})]^2 - 4\det(J(\bar{x}))}}{2}$$

Notice that this result is only valid when $\lambda \in (0, 1)$. When $\lambda = 1$, the determinant is undefined.

A.2 Proof of Proposition 4

We will show this result via Tarski's Fixed Point Theorem. We restrict attention to real-valued, increasing functions that map the domain $[0, b_u]$ into itself. Call this set $\mathcal{A}$. We define partial ordering over these functions, $\geq$, as follows: If $a_1, a_0 \in \mathcal{A}$, then

$$a_1 \geq a_0 \iff a_1(b_t) \geq a_0(b_t) \; \forall b_t \in [0, b_u]$$

It is easy to see that $(\mathcal{A}, \geq)$ is a complete lattice. The supremum and infinum of any subset of $\mathcal{A}$ must necessarily exist in $\mathcal{A}$ since it is bounded and closed.

We now define an operator, $T : \mathcal{A} \to \mathcal{A}$, as follows. Let $a_i \in \mathcal{A}$. Define a real-valued function q_i via the pricing recursion:

$$q_i(b_{t+1}) = \frac{[1 - g(b_{t+1})] \times [\lambda + (1 - \lambda) q_i(a_i(b_{t+1}))]}{1 + r}$$

This is a contraction for any positive r and thus there is a unique pricing schedule, q_i. It need not be continuous, since we never restricted a_i to be continuous. We can show that it is decreasing, but this will not be needed for the result.

Having defined q_i, we now define Ta_i as follows. Let $B_i(b_t) = \{b_{t+1} | q_i(b_{t+1})[b_{t+1} -$

$(1-\lambda)b_t] \geq \lambda b_t - s(b_t)\}$.

$$(Ta_i)(b_t) = \begin{cases} \inf B_i(b_t), & \text{if } B_i(b_t) \text{ is nonempty} \\ b_u, & \text{otherwise} \end{cases} \tag{B.1}$$

This operator dictates that the updated policy function meet the budget constraint whenever it is feasible to do so. In the event that this is not possible, it sets the it to the upper bound. We call the first case the *feasible* case and the second that *infeasible* case. Notice that Ta_i will not necessarily be continuous, even if a_i is. However, it will be the case that $Ta_i \in \mathcal{A}$, which is what matters.

To see this, note first that $(Ta_i)(b_t)$ will always be real-valued and in $[0, b_u]$ by construction. Further, it will be increasing by virtue of Lemma 1, since the commitment condition is satisfied by construction. This holds even in regions of infeasibility, since in all these regions the function is set to the upper bound. Thus, $T : \mathcal{A} \to \mathcal{A}$.

We now show that T is a monotone map. Suppose that we have two policies in $\mathcal{A}$: $a_1 \geq a_0$. We first show that $q_1(b_{t+1}) \leq q_0(b_{t+1})$. To see this, note that we can expand the pricing recursion into an infinite sum, in which case:

$$\begin{aligned} q_1(b_{t+1}) &= \frac{\lambda}{1-\lambda} \times \sum_{s=t}^{\infty} \left(\frac{1-\lambda}{1+r}\right)^{s-t+1} \Pi_{k=1}^{s-t+1}[1 - g(a_1^k(b_{t+1}))] \\ &\leq \frac{\lambda}{1-\lambda} \times \sum_{s=t}^{\infty} \left(\frac{1-\lambda}{1+r}\right)^{s-t+1} \Pi_{k=1}^{s-t+1}[1 - g(a_0^k(b_{t+1}))] \\ &= q_0(b_{t+1}) \end{aligned}$$

where $a_i^k(\cdot)$ is the application of the function $a_i(\cdot)$ k times. The inequality follows since $a_1^k(b_{t+1}) \geq a_0^k(b_{t+1})$ since both are increasing functions and since $a_1 \geq a_0$.

Now that we've established that the pricing schedules are ordered, there are essentially three cases that could arise:

1. Both $(Ta_1)(b_t)$ and $(Ta_0)(b_t)$ are feasible. In this case,

$$
\begin{aligned}
q_1(b^1_{t+1})[b^1_{t+1} - (1-\lambda)b_t] &\geq \lambda b_t - s(b_t) \geq 0 \\
\rightarrow q_0(b^1_{t+1})[b^1_{t+1} - (1-\lambda)b_t] &\geq \lambda b_t - s(b_t) \\
\rightarrow b^0_{t+1} &\leq b^1_{t+1}
\end{aligned}
$$

   In this case, we know that that q_0 is always bigger than q_1 for any b_{t+1}, but there are two possible cases. Either $q_0(b^1_{t+1})[b^1_{t+1} - (1-\lambda)b_t]$ is an increasing function, in which case the last inequality must hold since b^0_{t+1} is the infinum of that set; or $q_0(b^1_{t+1})[b^1_{t+1} - (1-\lambda)b_t]$ is a decreasing function, in which case we can attain the same revenue by shifting to the good, increasing side of the Laffer curve, $\hat{b}^1_{t+1} \leq b^1_{t+1}$. From here, we can apply the same argument as before to argue for the last line since $b^0_{t+1} \leq \hat{b}^1_{t+1}$.

2. Both $(Ta_1)(b_t)$ and $(Ta_0)(b_t)$ are infeasible. In this case we have that $(Ta_1)(b_t) = (Ta_0)(b_t) = b_u$.

3. Exactly one of $(Ta_1)(b_t)$ and $(Ta_0)(b_t)$ is feasible. In this case, we know that the feasible one must be $(Ta_0)(b_t)$, since the price is uniformly higher in this case. By the fact that $T : \mathcal{A} \rightarrow \mathcal{A}$, we will have that $(Ta_0)(b_t) \leq b_u = (Ta_1)(b_t)$.

In examining all three cases, it is clear that $Ta_1 \geq Ta_0$, and thus the operator is a monotone self-map. Applying Tarski's fixed point theorem tells us that the set of fixed points is a complete lattice, which implies that at least one exists and that they are all ordered according to $\geq$.

Now this does not tell us that a Markov Perfect Equilibrium exists, since a Markov Perfect Equilibrium will be a fixed point in which Ta exhibits feasibility for any b_t in Equation B.1. If we do have multiple fixed fixed points that are fully feasible in this sense, however, then the ordering result from Tarski's Fixed Point Theorem Applies and we will have Corollary 1.

This result also tells us the only conditions under which an equilibrium cannot exist. Equilibria can only not exist if there is some point of infeasibility in Equation B.1 i.e. if the primary deficit policy implies fiscal deficits that are too large to be feasibly financed and our monotone operator blows right past the region of feasibility as it iterates. Hence Proposition 4.

A.3 Proof of Proposition 7

Define a discrete and finite grid over debt levels, $\mathcal{B}$, and ensure that $0 \in \mathcal{B}$ as well us the upper bound, $b_u \in \mathcal{B}$. The upper bound implies a default probability of one. Let $B = |\mathcal{B}|$. We first define a numerical approximation to the MPE: A **Numerically Approximated Markov Perfect Equilibrium (NAMPE)** is a real-valued vector, $a \in [0, b_u]^B$, and a real-valued function $q(b_{t+1})$, such that

1. $q(b') = \frac{[1-g(b')]\times[\lambda+(1-\lambda)\times q(\hat{a}(b'))]}{1+r}$
2. $a_i = \inf \ \{b' |\ q(b')[b' - (1-\lambda)b_i] = \lambda b_i - s(b_i)\}$ for every index i on the debt grid.

where $\hat{a}(b')$ is a real-valued function constructed by linearly interpolating the vector a over the debt grid. Notice that for computational purposes, we will assume that the equilibrium exists and not build in the infeasibility contingency used to demonstrate existence and orderedness in Proposition 4.

To solve for these equilibria, we define an operator $T : [0, b_u]^B \to [0, b_u]^B$ i.e. a function that maps the set of N-dimensional real vectors into itself. This operator is defined as follows, given some initial $a \in [0, b_u]^B$:

1. Linearly interpolate a function $\hat{a}(b')$ to fill in the holes in the grid. We can then recursively define the function

$$q(b') = \frac{[1 - g(b')] \times [\lambda + (1-\lambda) \times q(\hat{a}(b'))]}{1+r}$$

   The real-valued function $q(\cdot)$ exists and is unique since the recursion is a contraction; monotonicity and discounting continue to hold with the addition of the linear interpolation.

2. For each point on the grid, $b_i \in \mathcal{B}$, define row i of Ta as follows:

$$(Ta)_i = \inf \ \{b' |\ q(b')[b' - (1-\lambda)b_i] = \lambda b_i - s(b_i)\}$$

   We assume for the purposes of numerical computation that the set of possible solutions is non-empty. Conditional on this assumption, notice that each $(Ta)_i$ must necessarily be in $[0, b_u]$, since revenue must always be raised at the auction and this is the Laffer curve's domain for new issuance. Thus, T maps $[0, b_u]^B$ into itself.

This vector operator behaves exactly as the functional operator in the proof of Proposition 4, and thus it is monotone on those regions of feasibility. Since we are not considering regions of infeasibility, we can also have that the operator T is continuous. This follows from the continuity of the underlying functions, $s(\cdot)$ and $g(\cdot)$.

We can now begin with the stability results. We will say the operator T **robustly converges from below (from above)** to $a^\star \in R^N$ if for some $a_0 << a^\star$ $(a_0 >> a^\star)$, $\lim_{n\to\infty} T^n a_0 = a^\star$. Suppose that we have two equilibria, a_L and a_H i.e. $Ta_L = a_L$ and $Ta_H = a_H$. Since T is a monotone operator, we will know that $a_L \leq a_H$.

We first show that T robustly converges to the lowest-debt equilibrium, a_L from below. Let $a_0 = 0^N$. Notice that in this case, we know that the pricing function only involves immediate default risk, which implies

$$q_0(b_{t+1}) = \frac{[1 - g(b_{t+1})] \times [\lambda + (1-\lambda)\bar{q}]}{1+r}$$

where $\bar{q}$ is the risk-free price, which is strictly positive. Turning to the budget constraint, since the sovereign gets a strictly positive price and must issue, we know that $Ta_0 \geq 0 = a_0$. We know that since T is monotone, then

$$a_0 \leq Ta_0 \leq T^2 a_0 \leq \dots$$

Further, we know that since $a_0 << a_L$, then for any n we have

$$T^n a_0 \leq T^n a_L = a_L$$

Thus, as n goes to infinity, $T^n a_0$ rises with each iteration. Since it is bounded above by a_L and is continuous, it must converge to something. Suppose that it converged to some $\hat{a}$ less than a_L, then it would have to be the case that in the limit, $T\hat{a} = \hat{a}$. But then a_L is no longer the lowest-debt equilibrium.

Thus, we know that when the low-debt equilibrium exists, we can compute it via a policy function iteration from below e.g. by starting with the zero price. We cannot generally say the same thing about the high-debt equilibrium from above. In fact, if there exists an $\hat{a} >> a_H$ such that $T\hat{a} \geq \hat{a}$, then T does **not** robustly converge to the high-debt equilibrium, a_H, from above or below. If it did, then there would have to be some $a_0 >> a_H$ such that $Ta_0 \leq a_0$ and (wlog) $a_0 << \hat{a}$. To see this, note that repeated iteration under this condition would imply

$$a_0 \leq Ta_0 \leq T^2 a_0 \dots$$

which by the same logic as before would eventually converge to a_H. But such an a_0 cannot exist. If it did, then we could apply Theorem 1 of Kostrykin and Oleynik (2012), which states that a monotone, continuous operator on an ordered vectored space implies a fixed point between the subsolution and the supersolution i.e. there must exist an $a^\star$ such that $a^\star \leq \hat{a}$ and $a^\star \geq a_0$ and $Ta^\star = a^\star$. But since $a^\star >> a_H$, this contradicts the fact that a_H is the high-debt equilibrium. Thus, we cannot robustly converge to the high-debt equilibrium from above.

To see below is even easier. Suppose that we started from some $a_0 << a_H$ but that $a_0 >> a_L$ (otherwise we know it would just converge to the low-debt equilibrium by the previous proposition). Then by the same argument as before, we will have that $\lim_{n\to\infty} T^n a_0 = a_L$. Thus, we would converge to the low-debt equilibrium, not its high-debt counterpart.

B Solution Algorithm

We know from Proposition 7 that iterative algorithms are unlikely to uncover the high-debt equilibrium in the presence of multiplicity. To do so will require an alternative method, which I propose here. The idea is to apply a shooting algorithm on the original dynamical system: Given b_t, we choose some q_t; the budget constraint then yields some b_{t+1}; q_{t+1} can then be chosen to satisfy the no-arbitrage condition of the lenders; the system is then ratcheted forward until a steady state is hopefully reached. Once there, we can use the implied trajectory to construct a guess for the policy and demand schedules.

The problem with such an algorithm is that in its current form it treats the sovereign as a price-taker, when in our equilibrium concept the sovereign can always choose the smallest of potential solutions for b_{t+1}. To impose this additional restriction, we will tweak the dynamical system in such a way as to obviate completely this contemporary multiplicity of solutions; in doing so, we can fearlessly apply the shooting algorithm described above and ratchet toward a solution that assumes such commitment powers. The solution to this approximating model can then be used as an initial guess in an iterative procedure on the original model. In practice, this algorithm tends to perform quite well in its ability to uncover the multiplicity of equilibria.

B.1 The Approximating Model

We begin by describing the approximating model.

$$\frac{s(b_t)}{\Delta} \; + \; q_t \times \left[b_{t+1/\Delta} - \left(1 - \frac{\lambda}{\Delta}\right) b_t \right] = \frac{\lambda}{\Delta} b_t \tag{C.1}$$

$$\begin{aligned} q_t = \frac{1}{1+\frac{r}{\Delta}} \times \Bigg[& \left[1 - g(b_{t+1/\Delta})\right] \times \left(\frac{\lambda}{\Delta} + \left(1 - \frac{\lambda}{\Delta}\right) q_{t+1/\Delta} \right) + \\ & g(b_{t+1/\Delta}) \times \left(0 + \left(1 - \frac{1}{\Delta}\right) q_{t+1/\Delta} \right) \Bigg] \end{aligned} \tag{C.2}$$

This approximating model essentially splits the original model into subperiods of length $1/\Delta$. Importantly, the Lifetime-Laffer Curve is preserved exactly i.e. if one were to compute the set of steady states in this model, they would be exactly the same as in the original model outlined in Section 2. In fact, that model is simply a special case of this one when $\Delta = 1$.

The benefit of this approximation is that for large Δ it lops off the Pareto-Dominated right-hand solution of the contemporaneous Laffer curve, thus guaranteeing the contemporaneous commitment. A proof of this can be found in the Online Appendix.

B.2 Algorithm

We begin by re-writing Equations C.1 and C.2 as follows:

$$b_{t+1/\Delta} = \left(1 - \frac{\lambda}{\Delta}\right) b_t + \frac{\frac{\lambda}{\Delta} b_t - \frac{1}{\Delta} s(b_t)}{q_t}$$

$$q_{t+1/\Delta} = \frac{q_t \left(1 + \frac{r}{\Delta}\right) - \left[1 - g(b_{t+1/\Delta})\right] \frac{\lambda}{\Delta}}{\left(1 - \frac{\lambda}{\Delta}\right) - \frac{g(b_{t+1/\Delta})}{\Delta}(1 - \lambda)}$$

How can this deconstruction be interpreted? Given some initial q_t, the budget constraint will give the required $b_{t+1/\Delta}$ required. Once we know $b_{t+1/\Delta}$, we must then rationalize the chosen q_t. This is done by adjusting the return in the future via $q_{t+1/\Delta}$ until the lenders' break-even condition is met.

This approach is used by Lorenzoni and Werning (2014), who explore a similar model in continuous time. The problem is that in the general model, as well as in their model, this method cannot guarantee contemporaneous commitment to debt issuance. With the approximating model, however, we know that only one solution exists for large Δ and that it is the 'good' one.

Thus, to solve the model, we simply apply a shooting algorithm to the approximating model for sufficiently large Δ. In practice, setting $\Delta = 20$ tends to work very well.

We consider two starting points, $b_0 = 0$ and $b_0 = b_u$. In each case, we employ a shooting algorithm over q_0, ratcheting the iterations forward. If they converge in a monotone way to a steady state, then we have found a solution to the approximating model for that Δ.

Our solution will consist of two sets of points: A set $\{(q_0, b_0), (q_{1/\Delta}, b_{1/\Delta}), (q_{2/\Delta}, b_{2/\Delta}), \dots\}$, all of which imply $b_t \geq \bar{b}$, and a set $\{(\hat{q}_0, \hat{b}_0), (\hat{q}_{1/\Delta}, \hat{b}_{1/\Delta}), (\hat{q}_{2/\Delta}, \hat{b}_{2/\Delta}), \dots\}$, all of which imply $\hat{b}_t \leq \bar{b}$. If the trajectories are monotone and contractionary around the steady state, then we can construct a point-wise estimates of the policy function and the demand schedule as follows:

$$a(b_t) = \begin{cases} b_{t+1/\Delta}, & b_t > \bar{b} \\ \hat{b}_{t+1/\Delta}, & b_t \leq \bar{b} \end{cases}$$

$$q(b_{t+1/\Delta}) = \begin{cases} q_t, & b_{t+1/\Delta} > \bar{b} \\ \hat{q}_t, & b_{t+1/\Delta} \leq \bar{b} \end{cases}$$

One could interpolate on these point-wise estimates to construct a numerical approximation to a Markov Perfect Equilibrium.

Further, if the approximating model and the general model share the same stability properties, we can use the solution to the approximating model to solve the general model. We can do so as follows. Denote the MPE solved using the shooting algorithm above as $a_\Delta(b_t)$ and $q_\Delta(b_{t+1/\Delta})$. We will solve the general model using an iterative algorithm. Let $a^i(b_t)$ and $q^i(b_{t+1})$ denote the estimate of the policy and price functions at iteration i.

We begin by setting $a^0(b_t) = a_\Delta(b_t)$ and $q^0(b_{t+1}) = q_\Delta(b_{t+1/\Delta})$. We then form a grid, $\mathcal{B}$ over the domain of b_t and proceed as follows:

1. For every $b_t \in \mathcal{B}$, solve $q^i(b_{t+1})[b_{t+1} - (1-\lambda)b_t] = \lambda b_t - s(b_t)$ for b_{t+1}. There will necessarily be multiple solutions for $i \geq 1$. Define

$$a^{i+1}(b_t) =$$
$$\min_{b' \in \mathcal{B}} \left(b' - \inf\{b_{t+1} | q^i(b_{t+1})[b_{t+1} - (1-\lambda)b_t] = \lambda b_t - s(b_t)\}\right)^2$$

2. For every $b_{t+1} \in \mathcal{B}$, define

$$q^{i+1}(b_{t+1}) = \frac{[1 - g(b_{t+1})] \times [\lambda + (1-\lambda)q^i(a^{i+1}(b_{t+1}))]}{1+r}$$

3. Continue until $\sup_{b' \in \mathcal{B}} |q^i(b') - q^{i+1}(b')| < \epsilon$ for some small ϵ.

This process generally converges to a solution, though it is not guaranteed to do so since it is not a contraction. Further, in the presence of multiple equilibria, initial guesses corresponding to distinct equilibria in the approximating model converge to distinct equilibria in the general model. This allows us to uncover multiplicity of equilibria, especially those equilibria that are normally unstable numerically.

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