LONG-TERM SOVEREIGN DEBT ISSUANCE UNDER LIMITED COMMITMENT *

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Abstract

I show that long-run beliefs can influence long-term sovereign debt prices and issuance under limited commitment by providing sufficient conditions for a multiplicity of stable steady states in a dynamic model. The mechanism requires only that the sovereign lack commitment to future debt issuance, not contemporaneous debt issuance. Novel policy implications are explored and a numerical example calibrated to Portugal illustrates how this multiplicity of steady states can give rise to multiple equilibria. The calibrated model also reveals that discrepancies in the speeds of convergence to the steady state can generate additional equilibrium multiplicity.

Keywords: Long-Term Sovereign Debt, International Finance, Limited Commitment, Long-Run Beliefs

1 Introduction

Markets for sovereign debt are ripe ground for sentiments. During many crisis episodes, and particularly during the recent crisis in the Peripheral Eurozone, many have spoken seriously if abstractly about the role beliefs play in generating or perpetuating crises. Summers (2000), Lane (2012), De Grauwe and Ji (2013a), and Wolf (2014) are just a few examples.

Most discussion of belief-driven crises centers around liquidity issues or debt of relatively short maturity (Cole and Kehoe, [1996], Rodrik and Velasco [1999], Bocola and Dovis (2015), or Aguiar et al. [2017]). This paper seeks to expand that discussion by exploring the unique role beliefs can play in the issuance of long-term sovereign debt, which tends to be immune to such liquidity issues (Chatterjee and Eyigungor [2012]).

In particular, I show that long-run beliefs can influence long-term sovereign debt prices and issuance under limited commitment. To do so, I construct a dynamic model of long-term sovereign debt with limited commitment and show that the model features multiple stable steady states. This multiplicity of steady states is generated along what I call the 'Lifetime-Laffer Curve,' which dictates the stationary sovereign debt auction revenue[1] generated by a given stationary debt level, factoring in prices that accurately

[1] I refer to 'auctions' here as Calvo (1988) i.e. a market in which there is a single sovereign borrower who issues debt and a group of lenders who purchase that debt competitively.

reflect default risk. Much like the contemporaneous 'debt-Laffer curve' of Calvo (1988), which plots debt issuance against the revenue generated at a given sovereign debt auction, the Lifetime-Laffer Curve can feature multiple stationary debt levels consistent with stationary revenue needs and for the same reasons: There will be one with low debt levels and high prices and another with the reverse, since increasing debt reduces prices.

Expectations regarding toward which of these solutions the economy will converge in the long run will impact the price schedule the sovereign receives in the initial period: If lenders anticipate that eventually debt levels will converge to the high-debt solution, they will offer a lower price as compensation for the greater associated long-run default risk i.e. dilution. The sovereign is forced to borrow more today to fill his fiscal deficit, which places him on the trajectory toward the high-debt solution, fulfilling lenders' expectations. On the other hand, if lenders anticipate debt levels to converge to the low-debt solution, they will offer a higher price since there is less dilution. The sovereign can then fill his fiscal deficit with less debt, placing him on the trajectory toward the low-debt solution and fulfilling lenders' expectations.

In keeping with the literature on sovereign debt and default (Eaton and Gersovitz [1981], Aguiar and Gopinath [2006], Arellano [2008], Hatchondo and Martinez [2009], Mendoza and Yue [2012], Aguiar et al. [2016]), I restrict attention to Markov Perfect Equilibria and assume throughout the analysis that the sovereign suffers from a limited commitment friction. In particular, I assume that the sovereign cannot commit to a debt issuance policy in future periods. In the initial period, he takes his own future actions as given. I do, however, assume that the sovereign has control over his issuance today. This assumption is in line with the aforementioned literature and distinguishes this work from Lorenzoni and Werning (2014) or Corsetti and Dedola (2013).

In this paper, I focus on coordination failures arising from the issuance of long-term sovereign debt itself. I thus abstract from the sovereign's consumption-saving decision and assume a fixed, albeit state-contingent, rule for the primary surplus in the vein of Leeper (1991), Woodford (2001), Schmitt-Grohé and Uribe (2007), Bi and Traum [2012]), or Bocola (2016). The idea here is to capture a debt auction as the relevant period, the duration of which tends to be too short to accommodate large adjustments in tax levels or spending. In related work, Stangebye (2016), I show that similar sentiment-driven dynamics still arise when this assumption is relaxed and the sovereign can actively adjust its primary surplus in response to belief fluctuations, though their behavior cannot be characterized as cleanly.

The analysis proceeds in three basic steps. First, I provide simple sufficient conditions that ensure multiple steady states exist that could serve as equilibrium limiting points. It is here that I outline and describe the Lifetime-Laffer Curve. I find that the parameter restrictions required to generate multiple stable limiting points are in the empirically relevant region e.g. a risk-free rate near zero, default probabilities near zero, and debt of sufficiently long maturity.

Second, I establish that the equilibrium set is necessarily ordered, which has robust policy implications. In a simple extension to the benchmark model, I show that third party programs initially designed to provide liquidity, such as the Clinton administration's guarantee of Mexican debt during the Tequila crisis or the Outright Monetary Transactions (OMT) Program during the Eurozone crisis, can have the arguably unintended consequence of preventing certain types of solvency defaults by coordinating beliefs on long-run outcomes. Roch and Uhlig (2014) demonstrate that these sort of programs can coordinate intraperiod beliefs in the case of liquidity crises while receiving an actuarially fair return. The logic here is similar: The third party, as a large, credible potential buyer can rule out the unpleasant, high-debt equilibrium without actually having to purchase debt: It can offer the lowest-debt equilibrium demand schedule for the foreseeable future, which will force private agents to coordinate on it. Essentially, it can lend its credibility to the sovereign to mitigate the limited commitment friction.

The fact that the equilibria are ranked imply that a benevolent third party would always *want* to intervene in this way. Further, and in contrast to Lorenzoni and Werning (2014), it will always be *able* to since there is no 'point of no return' i.e. a debt level beyond which equilibrium multiplicity vanishes. If the model features equilibrium multiplicity, it will do so for any debt level, and thus a third party could intervene at any point. The absence of a 'point of no return' is a novel implication of this model.

Third, I calibrate the model to Portugal during the recent crisis and show that these sorts of sentiment-driven fluctuations can have a massive impact on borrowing costs. I choose Portugal since my model naturally generates two puzzling features of the recent crisis in the Peripheral Eurozone. First, malignant market sentiments seemed to play a strong role. This is suggested in empirical work by De Grauwe and Ji (2013b) or Aizenman et al. (2013) and in more structural work by Corsetti and Dedola (2013) or Lorenzoni and Werning (2014). It is also suggested by the apparent success of the OMT liquidity provision program, which brought spreads down dramatically upon its announcement in the summer of 2012 but which has yet to have been called on by a member state (Altavilla et al. [2014]).

Further, most peripheral governments borrowed heavily throughout the crisis in the face of high spreads (Corsetti and Dedola [2013] or Conesa and Kehoe [2014]). This seems to contradict many models of both fundamental and sentiment-driven sovereign default, which generate the sharp prediction that sovereign borrowers ought to delever in the face of unexpected high spreads, not lever up (Arellano [2008] or Aguiar and Amador [2013]). However, it is completely consistent with Lifetime-Laffer Curve multiplicity, in which high spreads are both a cause and consequence of high borrowing.

The calibrated model suggests that sentiments can have little impact on equilibrium debt levels but massive impacts on borrowing costs. Debt levels are only 3.06% lower in the low-debt equilibrium steady state, which can be interpreted as a simple counterfactual, but borrowing costs as measured by the sovereign spread are 14.3 percentage points ($\approx$ 89%) lower. The reason for this discrepancy is the long maturity of the debt: Long-run beliefs can have an enormous impact on prices, but the translation from prices to debt levels will be muted since a relatively small fraction debt needs to be serviced at these drastically lower prices.

The calibrated model also reveals an additional source of multiplicity in the speed of convergence. There are two equilibria that converge to the same, high-debt steady state: The first converges slowly and the second quickly. The first features greater default risk earlier on than the second, and thus a worse pricing schedule in the initial period. Since the price schedule is worse, more borrowing is required and in earlier periods than in the second equilibrium, which leads to faster convergence. This is a novel feature of this model and suggests that even if long-run expectations remain consistently negative, market sentiments can still impact prices and debt levels in the short run. This finding accords with the weak correlation between fundamentals and spreads documented during the Eurozone crisis in De Grauwe and Ji (2013b).

Several authors have written about the impact of beliefs or sentiments in sovereign debt markets. Cole and Kehoe (1996) and Rodrik and Velasco (1999) argue that liquidity issues can plague short-term sovereign borrowers. Auclert and Rognlie (2016) find conditions for equilibrium uniqueness in the canonical model of Eaton and Gersovitz (1981), and Passadore and Xandri (2015) provide sufficient conditions for equilibrium multiplicity in a similar class of models. Both of these works pertain only to debt of short maturity; Aguiar and Amador (2016) and Stangebye (2016) provide numerical examples of equilibrium multiplicity and sunspot activity respectively in long-term sovereign debt markets as modeled by Chatterjee and Eyigungor (2012), but as of yet neither have formalized this result as is done here.

To the author's knowledge, the only others that have highlighted the role of dilution-driven coordination failures in long-term debt markets in the recent Eurozone crisis are Lorenzoni and Werning (2014). These authors also argue for a Laffer-curve type multiplicity in the spirit of Calvo (1988) with the explicit inclusion of long-term debt. In their environment, as in this paper, a dynamic lender coordination failure can place the economy on a malignant trajectory of high spreads and debt ratios. They term such events 'slow-moving crises.'

A key difference between their paper and this is that my equilibrium concept makes the stronger assumption that the sovereign can commit to both to its current primary deficit and to its current debt issuance, whereas they assume for tractability in their benchmark model that the sovereign can commit only to its current primary deficit. This assumption is crucial to many of the key novel results, such as the ranking of the equilibria and the absence of a 'point of no return' i.e. a debt level beyond which the equilibrium is unique. This last feature is particularly important since it implies a policy distinction: My model suggests that a third party could still come to the aid of a highly distressed and deeply indebted economy, whereas in their model it cannot. Their model also does not exhibit the additional source of sentiment dynamics derived from the speed of convergence.

The rest of the paper is divided as follows: Section 2 describes the model and formalizes the equilibrium concept; the primary analysis occurs in Section 3; Section 4 discusses the consequences of the results and policy implications; Section 5 provides a numerical application and discusses quantitative findings; and Section 6 concludes.

2 Model

I consider a model with two classes of agents. A unit mass of risk-neutral, deep-pocketed lenders and a single sovereign borrower with limited commitment. The sovereign issues a continuum of zero-coupon, long-term bonds with stochastic maturity, $\lambda \in (0, 1)$. In period t, the sovereign can commit neither to repay the debt in future periods, nor to a given debt issuance policy.

I restrict attention to equilibria that are Markov Perfect, in keeping with the literature on sovereign debt and default (Aguiar and Gopinath [2006], Arellano [2008], or Aguiar et al. [2016])

The sovereign is required to follow a bounded and time-invariant rule for the primary surplus,[2] $s(b_t)$, much like Leeper (1991) or Schmitt-Grohé and Uribe (2007).[3] I will assume that $s(b_t)$ is increasing, continuous, differen-

[2]See Stangebye (2016) or Aguiar and Amador (2016) for numerical examples of self-fulfilling crises in which the sovereign can shift his primary surplus in response to shifts in expectations.

[3]Fiscal rules instead interpreted as a constraint on a sovereign maximization problem have been explored by Hatchondo et al. (2015) and Alfaro and Kanczuk (2017).

tiable, and bounded. Many real-world fiscal institutions, such as those in the Maastricht treaty, place strong restrictions on fiscal policy as a function of the economy's current state.

However, the sovereign can choose how much debt to issue, b_{t+1}, in period t. In fact, he is a monopolist in his own debt market and faces a demand schedule $q(b_{t+1})$. The key friction is his inability to commit, at time t, to b_{t+s} for $s \geq 2$.

He seeks to minimize the net present value of the face value of debt issued over the course of his life. Given his inability to commit to debt issuance in future periods, this problem is most intuitively expressed recursively

$$M(b_t) = \min_{b_{t+1}} \; b_t + \frac{1}{1+r} M(b_{t+1})$$

$$\text{s.t.} \qquad s(b_t) + q(b_{t+1})[b_{t+1} - (1-\lambda)b_t] \geq \lambda b_t \tag{1}$$

where $M(b_t)$ is the net present value of debt stream at time t and r is the risk-free rate.

Given that the fiscal process is predetermined as a function of current states, one might think of this as the objective function of the treasury, who seeks to minimize the debt issuance required to finance a given stream of primary deficits determined by some other body e.g. the legislature. This problem is generally non-trivial, as Calvo (1988) shows there can be multiple solutions. It is worth noting that similar results hold if we allow for the primary surplus to respond to shifts in prices or expectation shifts; I show this in related work (Stangebye [2016]).

There will be another source of limited commitment, which is the inability of the sovereign to commit to repayment. While this is important, it is not the chief investigation of this paper. Consequently, I will assume default takes a relatively simple form. There is a time-invariant outside option associated with default, $\tilde{X}$, which is an iid continuous random variable distributed on $[0, b_u]$ for some $b_u < \infty$. The sovereign will default in time t if the debt level, b_t strictly exceeds $\tilde{X}$. For simplicity of exposition, I define $g(b_t) = E_{\tilde{X}}\left[\mathbf{1}\{b_t > \tilde{X}\}\right]$. Without loss of generality, I will take the function $g(b_t)$ as a primitive and assume it to be continuous, differentiable, increasing and bounded between zero and one.

On the other side of the market, risk-neutral, deep-pocketed foreign lenders solve a portfolio allocation problem involving the risk-free asset and risky sovereign debt. A no-arbitrage condition is sufficient to derive a Markov Perfect equilibrium pricing expression:[4]

$$q(b_{t+1}) = \frac{[1 - g(b_{t+1})][\lambda + (1-\lambda)q(a(b_{t+1}))]}{1+r} \tag{2}$$

[4]See Chatterjee and Eyigungor (2012) or Hatchondo and Martinez (2009) for details on the derivation of such a condition.

Since the debt is long-term, the return on an investment in risky sovereign debt is dependent both on the repayment of that debt in the next period and its future price in secondary markets, $q(a(b_{t+1}))$, where $a(\cdot)$ is the equilibrium debt issuance policy implied by Recursion 1.

2.1 Equilibrium Concept

Definition 1. *A **Markov Perfect Equilibrium (MPE)** is a pair of time-invariant functions, $q(b_{t+1})$ and $a(b_t)$, such that*

1. *$b_{t+1} = a(b_t)$ solves Recursion 1 given debt demand $q(\cdot)$.*
2. *$q(b_{t+1})$ satisfies Recursion 2 given the issuance policy $a(\cdot)$.*

A Markov Perfect Equilibrium will be said to be **Monotone** if, in addition, $a(b_t)$ is an increasing function. I will follow Chatterjee and Eyigungor (2012) and restrict attention to these equilibria since they will generate more intuitive and tractable dynamics.

The set of equilibria is potentially much larger than our Monotone Markov restriction imposes (see Passadore and Xandri [2015]), but given that multiplicity will still arise under this restriction I impose it for tractability.

2.2 Commitment to Contemporaneous Debt Issuance

Before analyzing the model in depth, I discuss briefly what the model's assumptions imply intuitively. Recursion 2 is a relatively straightforward interpretation of the pricing expression in models such as those in Chatterjee and Eyigungor (2012) or Hatchondo and Martinez (2009) in our environment. The key thing to note here is that the revenue function implied here i.e. $q(b_{t+1})[b_{t+1} - (1-\lambda)b_t]$ features a 'Laffer curve' of the sort described first by Calvo (1988). It is a continuous function that is equal to zero both at $b_{t+1} = (1-\lambda)b_t$ and at $b_{t+1} = b_u$. The reason behind the former is trivial, since there is no issuance in this case; the reason behind the latter is the following: If the sovereign chooses to issue b_u, the price will necessarily be $q(b_u) = 0$ regardless of the borrowing policy function that applies in the future since this ensures default tomorrow.

Thus, if revenue is to be gained via issuance it must mean that, generically, there are at least two ways of raising a given amount of revenue. One on the increasing portion of the revenue function and another on the decreasing portion. A heuristic example can be found in Figure 1, in which the sovereign can satisfy his revenue requirements either by issuing around $b_{t+1} = .74$ or $b_{t+1} = .98$.

Given that the sovereign is allowed to choose this and seeks to minimize

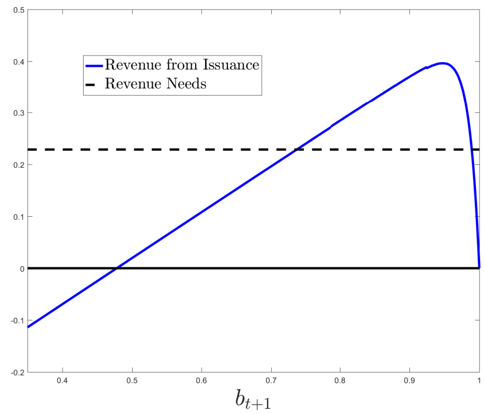


Figure 1: The Contemporaneous Debt 'Laffer Curve'

the net present value of debt levels, he will always issue on an increasing portion of this function i.e. the smallest possible level of debt that satisfies the budget constraint. This assumption is more in keeping with the quantitative literature spawned from Eaton and Gersovitz (1981) than the similar models of Lorenzoni and Werning (2014) or Corsetti and Dedola (2013), in which the sovereign cannot control on which region of the domain of the revenue function he is.

Despite this ability, which I will call **Commitment to Contemporaneous Debt Issuance**, he still lacks the ability to commit to debt issuance *in the future*. This will be important, since it will be a necessary condition for equilibrium multiplicity when it exists. As I will show, without this restriction, he could always choose to commit to the policy function of the lowest-debt equilibrium and thereby strictly improve his objective function.

3 Analysis

I now proceed to analyze the properties of Monotone Markov Perfect Equilibria. For simplicity of exposition, all lemmas and theorems take as given all assumptions listed prior their statement.

I show first that Markov Perfect Equilibria exhibit steady state dynamics.s

Lemma 1. *Any Monotone Markov Perfect Equilibrium converges monotonically to a steady state. Further, if every potential steady state debt level, $\bar{b}$, satisfies the additional* ***Global Convergence Condition***

$$[r + \lambda\bar{b}g'(\bar{b})]\left[1 - \frac{s(\bar{b})[1-\lambda]}{\lambda\bar{b}}\right] < rs'(\bar{b}) + [1 - g(\bar{b})]$$

then any Monotone Markov Perfect Equilibrium converges monotonically to the same steady state from above and below.

Proof. See Appendix A. □

As with the rest of our assumptions, I will assume the global convergence condition for the remainder of the analysis.

The global convergence condition ensures that, in equilibrium, $a'(b_t)$ is not too large. It will tend to be satisfied when four objects are small: $r, \lambda, g(\bar{b})$, and $g'(\bar{b})$. These all ensure that marginal price changes never become so large that the sovereign must compensate with excessive marginal changes in its borrowing behavior.

This condition is not difficult to satisfy. With long-term debt and spreads nearer to zero than one, this is the empirically relevant region. I will give a calibrated example in Section 5 in which this condition holds.

Lemma 1 is especially useful since I will soon establish that there can be a multiplicity of candidate equilibrium limiting points. To do so, I first define carefully what that means.

Definition 2. *A* ***candidate equilibrium limiting point*** *is a stable steady state with real-valued characteristic roots.*

Stability is a very intuitive requirement for a limiting point to satisfy. Real-valued roots simply dictate that equilibrium objects converge to the steady state monotonically instead of spiralling around it, since this sort of spiralling is inconsistent with Theorem 1.

Let $\mathcal{B}_l$ be the set of candidate equilibrium limiting points. I now provide conditions under which it has at least two elements. To begin, I will require a couple more assumptions.

Assumption 1. $\lim_{b \to 0} s(b)/b < \frac{r}{r+\lambda}$.

Assumption 2. $s(b_u)/b_u < \lambda$.

These assumptions place upper bounds on the average response of the surplus to debt levels. Together, they will imply that in any steady state, some debt issuance will be required. Even at high debt levels, the sovereign will not respond with surpluses so strongly as to obviate the need to issue new debt to roll over existing debt. Together with the next assumption, they will ensure a set of stationary points that could serve as steady states.

Assumption 3. $\exists \hat{b} \in (0, b_u)$ *such that* $\frac{s(\hat{b})}{\hat{b}} > \left[1 - \frac{\lambda[1-g(\hat{b})]}{r+g(\hat{b})+\lambda[1-g(\hat{b})]}\right] \lambda$

In words, while it needs to be small at the boundaries, the average surplus should be sufficiently large at some interior point. All together, these deliver the following proposition:

Lemma 2. *There exist at least two steady state solutions that satisfy both the lenders' pricing condition and the sovereign's budget constraint.*

Proof. This is shown by constructing an object that I will term the **Lifetime-Laffer Curve**. I construct it by restricting attention to stationary points. Evaluating Recursion 2 at any constant level of debt, $\bar{b}$, implies a constant price given by

$$\begin{aligned} \bar{q} &= \frac{[1-g(\bar{b})][\lambda + (1-\lambda)\bar{q}]}{1+r} \\ \implies \bar{q}(\bar{b}) &= \frac{\lambda[1-g(\bar{b})]}{r+g(\bar{b})+\lambda[1-g(\bar{b})]} \end{aligned} \tag{3}$$

Combining this with the sovereign's steady state budget constraint implies

$$\begin{aligned} s(\bar{b}) + \bar{q}(\bar{b})\lambda\bar{b} &= \lambda\bar{b} \\ \rightarrow LL(\bar{b}) = \frac{1}{\lambda}\frac{s(\bar{b})}{\bar{b}} + \bar{q}(\bar{b}) &= 1 \end{aligned} \tag{4}$$

$LL(\bar{b})$ sketches out the **Lifetime-Laffer Curve**. The left-hand-side is a measure of the excess (or deficit) auction revenue generated at a given candidate stationary debt level. When Equation 4 holds, then stationary revenue needs are exactly met by stationary debt issuance.

The existence of $\hat{b}$ is merely a strong feasibility condition. If no such $\hat{b}$ existed in which the inequality held weakly, then the system would have no steady states. Under the feasibility condition, the following must hold.

1. Since $\lim_{\bar{b}\to 0} s(\bar{b})/\bar{b} < \frac{r}{r+\lambda}$, then $\lim_{\bar{b}\to 0} LL(\bar{b}) < 1$. Since $LL(\cdot)$ is continuous, by the intermediate value theorem, there exists some $b_L \in (0, \hat{b})$ such that $\frac{1}{\lambda}\frac{s(b_L)}{b_L} + \bar{q}(b_L) = 1$.

2. Since $\frac{s(b_u)}{b_u \lambda} < 1$ and $q(b_u) = 0$, then $LL(b_u) < 1$, and consequently by the intermediate value theorem there exists some $b_H \in (\hat{b}, b_u)$ such that $\frac{1}{\lambda}\frac{s(b_H)}{b_H} + \bar{q}(b_H) = 1$.

□

The intuition behind Lemma 2 is the same as Calvo's (1988) i.e. the excess auction revenue function exhibits an inverted-U shape: As stationary debt levels increase, initially revenue levels increase and we arrive at a sustainable point. As we increase it further, however, stationary prices fall. Eventually prices (and thus revenue) go to zero, and on its downward trajectory we find another sustainable point. The conditions on the surplus function simply imply that this effect is not offset by the response of the primary surplus.

One can see this result graphically in Figure 2, in which there are clearly exactly two solutions: A low-debt solution and a high-debt one.[5] In the low debt solution, the sovereign's budget constraint is satisfied since he gets a high steady state price and thus can issue a low amount of debt; in the high

[5]Figure 2 will come from the calibrated example in Section 5, which will feature a linear surplus rule and a convex default rule.

debt solution, the opposite is true i.e. the steady state price is low and thus the sovereign must issue more debt.

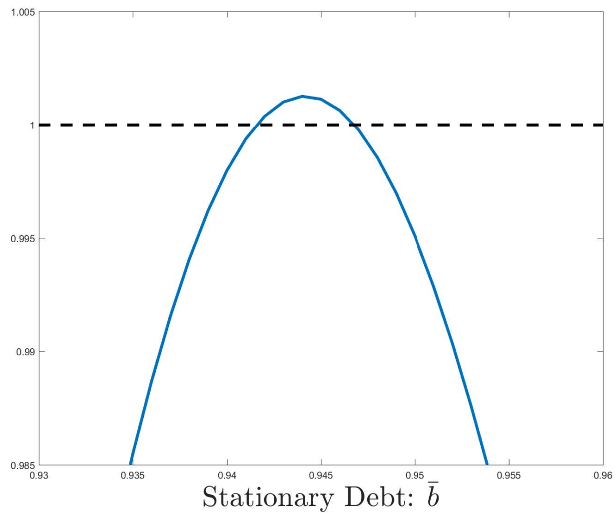


Figure 2:
Multiplicity via the Lifetime-Laffer Curve

In order to determine whether these steady states can be interpreted as limiting points for a Markov Perfect Equilibrium, I first clarify the role of a Markov Perfect Equilibrium in governing the dynamics of prices and debt levels.

Lemma 3. *Any Monotone Markov Perfect Equilibrium describes a solution* *to the following dynamical system:*

$$b_{t+1}(b_t, q_t) = \frac{\lambda - s(b_t)}{q_t} + (1-\lambda)b_t$$
$$q_{t+1}(b_t, q_t) = \frac{1}{1-\lambda}\left[\frac{(1+r)q_t}{1 - g(b_{t+1}(b_t, q_t))} - \lambda\right]$$

whose steady states are those described in Lemma 2.

Proof. A solution to the dynamical system is a prescription for the jump variable, q_t, in terms of the state variable, b_t. I denote this with $q_t(b_t)$. Following this initial assignment, the system then ratchets forward. From an MPE, one can define the solution as $q_t(b_t) = q(a(b_t))$. This will imply that $a(b_t) = b_{t+1}(b_t, q_t(b_t))$, which is internally consistent.

This is feasible since an MPE satisfies these equations by construction: The first is the budget constraint and the second is the pricing condition. The minimization in Recursion 1 works to select the initial q_t. □

Note that the reverse of Lemma 3 is not true: Once cannot construct a Markov Perfect Equilibrium from only a solution to the dynamical system. This is because a Markov Perfect Equilibrium requires more information than the dynamical system provides, in particular the price and borrowing behavior off of the trajectory implied by the equilibrium path.

We now know that a Monotone Markov Perfect Equilibrium converges to a steady state and that there are a multiplicity of such steady states associated with an intuitive but non-linear dynamical system. To connect these dots, I now endeavor to explore the local properties of these steady states. We are chiefly interested in two properties: Non-spiral dynamics (real-valued characteristic roots) and stability.

I begin with non-spiral dynamics. A steady state with complex roots cannot serve as a limiting point for a Monotone Markov Perfect Equilibrium, since Theorem 1 states that it must converge *monotonically* to its steady state. I begin with our assumptions, for simplicity of exposition:

Assumption 4. *The steady states from Lemma 2 satisfy either*

1. $1 - \lambda + \frac{\lambda - s'(\bar{b})}{q(\bar{b})} \leq 0$

2. $\frac{(1+r)g'(\bar{b})\lambda\bar{b}}{[1-\lambda][1-g(\bar{b})]^2}$ *is sufficiently small.*[6]

Lemma 4. *The steady states from Lemma 2 locally exhibit non-spiral dynamics.*

Proof. See Appendix A. □

[6] I define 'sufficiently small' very precisely in the proof of Lemma 4.

Notice that Assumption 4 is similar to those required for equilibrium convergence in Theorem 1 i.e. it will tend to hold when $r, \lambda, g(\bar{b})$, and $g'(\bar{b})$ are sufficiently small, which is the empirically region. Consequently, it is relatively easy to satisfy.

Being rid of spiral dynamics, the next step is to ensure local stability. Conveniently, it can be boiled down to a simple lower bound on the derivative of the surplus function

Assumption 5. *The steady states from Lemma 2 satisfy the following inequality:* $s'(\bar{b}) > \lambda - \frac{\lambda[1-g(\bar{b})]}{r+g(\bar{b})+\lambda[1-g(\bar{b})]}\left(1 + \frac{1+r}{[1-\lambda][1-g(\bar{b})]}\left[\lambda\bar{b}\frac{g'(\bar{b})}{1-g(\bar{b})} - 1\right]\right)$

Lemma 5. *The steady states from Lemma 2 are locally stable.*

Proof. See Appendix A □

Putting all of these assumptions and results together, we finally arrive at our principle result.

Theorem 1. *There are at least two elements in the set of candidate equilibrium limiting points,* $\mathcal{B}_l$*.*

Proof. This follows from Lemmas 2 through 5. □

4 Discussion

4.1 Commitment and Multiplicity

It is natural to wonder whether the sovereign's ability to choose debt issuance contemporaneously obviates any potential multiplicity of equilibria generated by a multiplicity of steady states. Interestingly, this is not the case. I will show this explicitly in a calibrated example, but the intuition for is easy to understand.

Suppose that Theorem 1 holds and we denote the high-debt steady state by b_H. Now suppose the sovereign follows any policy, $\hat{a}(\cdot)$, such that $\lim_{n\to\infty} \hat{a}^n(b_{t+1}) = b_H$ for any $b_{t+1} \in [0, b_u)$. In period t, the sovereign cannot commit to change this policy tomorrow. All he can do is choose b_{t+1} today. But regardless of which b_{t+1} he chooses, in the limit he will still travel to b_H over time. Thus, if $\hat{a}(\cdot)$ constituted part of an equilibrium, the ability to choose b_{t+1} would not eliminate this equilibrium because of the sovereign's inability to commit to b_{t+s} for any $s > 1$.

However, if the sovereign *could* commit to b_{t+s} for any $s > 1$, the multiplicity would disappear.[7] This is because the sovereign could (and would)

[7] The assumption here is that the sovereign can credibly commit to a trajectory of issuance, even if

commit to the best possible Monotone MPE i.e. if $a_L(\cdot)$ is the best equilibrium[8], then $b_{t+n} = a_L^n(b_t)$ for any $n > 0$. This necessarily would converge to $b_L < b_H$, thus eliminating multiplicity arising from multiple steady states.

4.2 Assumptions

Though many distinct assumptions must hold to ensure Theorem 1, the region in the parameter space that they outline is non-degenerate. I will show this explicitly with a calibrated example in Section 5, but intuitively these assumptions will tend to hold when $r, \lambda, g(\bar{b})$, and $g'(\bar{b})$ are sufficiently small. This is the empirically relevant region e.g. low interest rates, default frequencies nearer to zero than one, and long-maturity debt.

4.3 Policy Implications

Having established a potential source of belief shifts via Lifetime-Laffer Curve multiplicity, I now explore what this could imply for policy. A final assumption is required.

Assumption 6. $\lambda b_t - s(b_t)$ *is a strictly positive function of* b_t*.*

Under this assumption the sovereign never engages in active debt buybacks. He may delever, but it will be by letting existing debt mature and

it cannot commit to repayment.

[8] I will provide conditions in Section 4.3 under which a best equilibrium exists.

not rolling it over rather than actively repurchasing debt. This is not an unrealistic assumption: Bulow and Rogoff (1988) argue that debt buybacks would never be optimal in a similar environment in which the sovereign can choose the surplus. And in the case of the Eurozone debt crisis, none of the peripheral economies in crisis engaged in buybacks until the crisis was over.

We also need to restrict mildly the equilibrium set, but not in a way that binds in practice.

Definition 3. *A Monotone Markov Perfect Equilibrium is said to be **bounded from below** if and only if for all* $b_t \in [0, b_u]$, $q(a(b_t)) \geq q_{min}$, *where* $q_{min} = \frac{\sup_b s'(b) - \lambda}{1 - \lambda}$.

The additional restriction on the realized equilibrium price, which is tantamount to a cap on the equilibrium spread, works to maintain the monotonicity restriction across different Markov Equilibria. This price restriction is never binding in any of the numerical results in Section 5.

I can now establish an orderedness result:

Theorem 2. *If any Monotone Markov Perfect Equilibrium bounded from below exists, then a 'best' Monotone Markov Perfect Equilibrium bounded from below,* $(a_L(b_t), q_L(b_{t+1}))$, *exists i.e. if* $(a_H(b_t), q_H(b_{t+1}))$ *is any other* *equilibrium, then* $a_L(b_t) \leq a_H(b_t)$ *for all* $b_t \in [0, b_u)$ *and* $q_L(b_{t+1}) \geq q_H(b_{t+1})$ *for all* $b_{t+1} \in [0, b_u)$.

Proof. See Appendix A □

This result formalizes the notion that negative long-run beliefs are always things to be avoided and thus are the sort of thing policymakers should be worried about. This is non-trivial with long-term debt, since low prices can be advantageous during debt buybacks, which is why we require Assumption 6.

Now suppose that we had two such equilibria, with one converging to b_H and the other converging to b_L. If both are bounded from below, we now know that the former will be Pareto-dominated by the latter. Consequently, this is the sort of issue that a policymaker may wish to resolve.

Since Lifetime-Laffer Curve multiplicity is fundamentally an issue of coordination and beliefs, liquidity provision from a large, credible third party such as a central bank can in fact eliminate the high-debt equilibrium. I take the third party to be a deep-pocketed actor with the capacity to purchase sovereign debt at a price of its choice in any period. It offers the sovereign an alternate, time-invariant demand schedule, $q_{TP}(b_{t+1})$. Importantly, unlike the sovereign, the third party is assumed to be able to commit to future actions. I will not specify the cost of doing so, since it will not be relevant as long as it can commit.

In any period, the sovereign has the option to issue either to private creditors at the equilibrium price schedule, $q(\cdot)$, or to the third party. Under this assumption, the following holds.

Corollary 1. *A large, credible third party can costlessly engender the Pareto-optimal allocation at any initial debt level, b_t.*

Proof. The central bank can offer a demand schedule

$$q_{TP}(b_{t+1}) = q_L(b_{t+1})$$

where $q_L(b_{t+1})$ is the lowest-debt equilibrium pricing schedule. If it does this, the sovereign can always choose to borrow from the third party into the infinite future. If it were to borrow from the third party at this rate, it would imply an upper bound on the debt policy given by $a_L(b_{t+1})$ in every period after t. This implies, that private creditors can break even by offering the low-debt equilibrium price. Since they are competitive, they do so. Given this, the sovereign can borrow from private creditors instead of the third party and the third party does not need to purchase anything. $\square$

Since limited commitment is the problem, credibility is the solution. Just like the provision of deposit insurance in Diamond and Dybvig (1983), it does not need to be the case that the sovereign gains credibility to solve the problem; rather, the central bank or another credible third party, can lend the sovereign its credibility.

Thus, the provision of liquidity can prove effective in the case of a crisis of this nature, *even though all defaults are fundamental and not driven by liquidity concerns.* Notice that liquidity provision here is not interpreted as a bailout or a rescue package. Even in the low-debt trajectory, there is accurately priced default and dilution risk, and the the third party is assumed to *not* come to the rescue.

This is in many senses true to the way that, for example, the OMT in the Eurozone is actually set up. Stringent conditions need to be met for a member state to apply for OMT, and these conditions are least likely to be met when the sovereign is experiencing a crisis (Wolf [2014]). For instance, in recent years turbulent Greece has not met the conditions to apply for OMT help since they have not managed to regain complete access to private lending markets, which is a necessary condition of the OMT program.

Thus, the model predicts that providing liquidity in this way may not be an effective tool for preventing all future defaults (at least not costlessly), but it *can be* an effective way to prevent a malignant, sentiment-driven build-up of debt and spreads. Further, and in contrast to Lorenzoni and Werning (2014), the central bank is always *able* to do this. This is because equilibrium multiplicity pervades for all initial debt levels, whereas in their model there is a 'point of no return' i.e. a debt level beyond which the equilibrium is unique and liquidity provision is thus ineffective.

This result also stands in contrast to related work (Stangebye [2016]), where I find that this sort of liquidity provision cannot always be used effectively. The reason for the discrepancy is simple. In that model, a 'good' equilibrium may not exist. Abnormally high prices propped up by sentiments may be a result of an abnormally low default value, which happens to be low because 'bad' beliefs prevail following the default. In contrast, in the model presented here, a good equilibrium always exists, and thus intervention can always be effective.

Which conclusion to draw for a given scenario would likely depend on the circumstances; this model assumes a very rigid fiscal process that cannot adjust to debt prices or expectation shifts, whereas Stangebye (2016) assumes that the fiscal authority can respond to price changes or shifts in expectations more generally. The former is likely a better fit for Eurozone economies, while the latter perhaps better characterizes emerging economies more broadly (Aguiar et al. [2016]).

5 Quantitative Analysis

I now show that the multiplicity implied by the Lifetime-Laffer Curve can translate to a multiplicity of Markov Perfect Equilibria. I begin with a brief description of the solution method and then provide numerical examples.

5.1 Solution Method

To uncover a multiplicity of equilibria, I apply a novel technique. I use a finite-difference solution technique on an approximating model to find a good initial guess for an iterative algorithm. Starting from any b_0, I can apply the shooting algorithm over q_0 and construct trajectories toward one of the limiting points implied by the Lifetime-Laffer Curve. The solution to the approximating model provides a demand curve for debt that can be used as an initial guess in an iterative algorithm to solve the benchmark model. Details regarding the approximating system and solution technique can be found in Appendix B.

5.2 Calibrated Example

To explore its quantitative scope, I calibrate the model to the recent crisis in the Peripheral Eurozone.

The example provided is meant to be illustrative rather than an estimation. Thus, for simplicity I suppose that the surplus function takes the parametric form $s(b) = \kappa_0 + \kappa_1 b$. This is the popular form employed by Leeper (1991), Schmitt-Grohé and Uribe (2007), and Bocola (2016) among others. Also for simplicity, I assume a default probability given by $g(b) = b^\alpha$ for all $b \in [0, b_u]$, where $b_u = 1$.

The parameters of the model are roughly chosen to match the experience of Portugal. The exercise could be performed for any of the distressed Eurozone countries at the time since most of them exhibited a prolonged period of borrowing into high spreads. Portugal, however, fits particularly well because in addition to this debt levels stayed relatively high in the years following the crisis i.e. after the supposed expectations shift. This is consistent with numerical predictions that these coordination failures have a much larger impact on spreads than debt levels.

I will assume that $\kappa_1 = 0.5$, as Bocola (2016) does for Italy. The average maturity will accord with the data at 5.74 years i.e. $\lambda = 0.0437$. I will set the risk-free rate to be very low in accordance with its levels during the crisis i.e. $r = 0.0025$.

α and κ_0 are chosen jointly to both match a stationary high-debt equilibrium annual bond yield of 17%, which is slightly above the peak yield in mid-2012; and to ensure that two solutions exist along the Lifetime-Laffer Curve. I aim to find the lowest possible decile of α such that this is possible, since many different pairs of α and κ_0 will work. This exercise delivers $\alpha = 70$ and $\kappa_0 = -.4569$. Such severe convexity in the default function is typical in such models of endogenous default as Arellano (2008) or Aguiar and Gopinath (2006), in which default probabilities are near zero for almost the entire debt domain and spike near the endogenous credit limit.

5.2.1 Steady States

The parameters and exogenous functions satisfy all of the assumptions outlined in Section 3, and thus Theorems 1 and 2 hold.[9] I derive the Lifetime-Laffer Curve found in Figure 2. Here, there are exactly two solutions: $b_L = 0.9244$ and $b_H = 0.9536$. This implies that debt levels are 3.16% higher in the high-debt equilibrium. The concomitant steady-state prices exhibit larger differences, with $q_L = 0.8687$ and $q_H = 0.5221$. The former implies a stationary annual spread of 1.7% while the latter implies 16.0%. Recall that the latter was calibrated, so the former can be considered a

[9] Justification of this assertion can be found in Appendix C, along with the characteristic roots of the limiting points.

counterfactual.

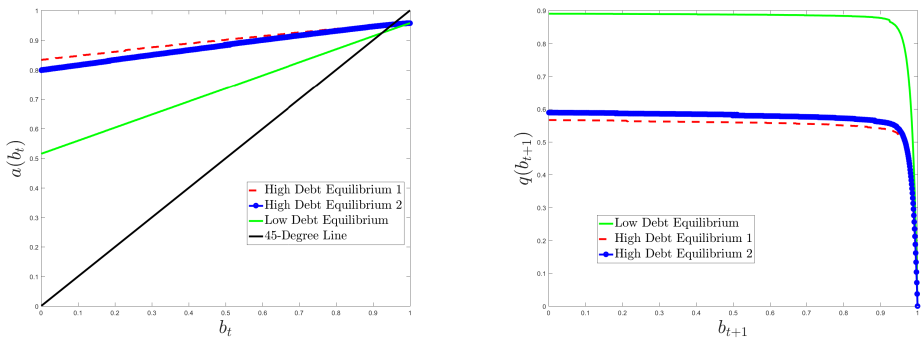
The example thus suggests that for reasonable parameters, the impact of equilibrium selection on debt levels is quite small but the impact on yields and borrowing costs is enormous. This is for two reasons. First, the debt is of relatively long maturity, which implies that little of it needs to be rolled over at any given auction. This mitigates the impact of spreads on debt build-up. Second, the default function is very convex, so small changes in equilibrium debt levels can have large effects on default probabilities and, consequently, borrowing costs.

5.2.2 Equilibrium Analysis

Having fully parameterized the model, I now compute and analyze the equilibria.

Result 1. *There are at least three Monotone Markov Perfect Equilibria bounded from below. The least of them converges to the low-debt steady state, while the others converge to the high-debt steady state.*

Solving the model quantitatively using the algorithm outlined in Appendix B reveals that both steady states can act as limiting points: There is one MPE that converges to b_L and two others that converge to b_H. This is particularly interesting since the low-debt equilibrium is the only one that is numerically stable under policy function iteration. There may in fact be more equilibria; these are only the ones uncovered by my algorithm.



(a) Policy Functions

(b) Price Schedules

Figure 3:
Markov Perfect Equilibria: Calibrated Example

The policy functions and price schedules associated with some of these equilibria can be found in Figure 3. Note that the equilibria are ordered, which accords with Theorem 2. The low-debt equilibrium converges to a lower debt level in the long run, and thus there is less associated dilution and a greater price than its high-debt counterparts. The reason for the discrepancy *across* high-debt equilibria i.e. the basis for indeterminacy is more subtle and revolves around the *speed of convergence*: High-Debt Equilibrium 1 converges to b_H more quickly than High-Debt Equilibrium 2, which implies greater dilution and a worse price. This can be seen in Figure 4. This worse price causes the need to issue more at low debt levels in High-

Debt Equilibrium 1 than in High-Debt Equilibrium 2. But this excessive issuance causes faster convergence to b_H in High-Debt Equilibrium 1, and thus the dynamics are self-fulfilling.

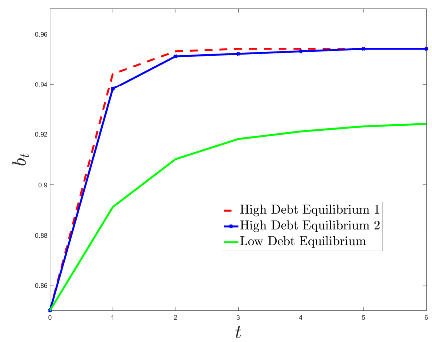


Figure 4:
Markov Perfect Equilibria: Debt Trajectories

5.2.3 Comparison with Lorenzoni and Werning (2014)

Many of these results differ from those of Lorenzoni and Werning (2014), who explore a similar model. The assumptions made in this model are more restrictive e.g. I explicitly assume that the sovereign has the ability to commit to debt issuance in any period and not just revenue; but the environment is also different e.g. the *arrival rate* of a default depends on the debt level and not just the probability of default conditional on some exogenous arrival. Since the models are quite different and many different assumptions could be driving the discrepancies, a brief discussion of the key differences is in order.

First, in their model a sort of a Lifetime-Laffer Curve exists, in which there are two steady states. In both models, the low-debt steady state is saddle-path stable. However, in their model the high-debt steady state is always unstable, while in this model, it is *globally stable*. Thus, their high-debt equilibrium does not traverse toward the high-debt steady state, as it does in this model; rather, debt and yields ratchet off toward infinity in a non-stationary way.

The difference in the local stability properties is *not* a result of the assumptions regarding commitment, since local stability can be computed before the global equilibrium is defined with the additional commitment restrictions. Rather, it is a product of disparate model assumptions, particularly their assumption that there is a 'cap' on the primary surplus $\bar{s}$, regardless of the debt level. Thus, as debt levels grow large, eventually the primary surplus stops attempting to reverse course and debt levels begin to explode. That is not the case in our model, in which there is no cap on the primary surplus. In fact, the primary surplus here is always increasing and thus grows *even larger* as debt levels grow. This contractionary force pushes large debt levels back toward the high-debt steady state, lending it stability.

This stability is likely what lends my model the additional multiplicity channel coming through the speed of convergence to the steady state, which is also not present in their model.

Second, in this model there is no 'point of no return' i.e. a debt level beyond which the equilibrium is unique. In this model, long-run beliefs are influential at *any* debt level, while in their model beliefs can only matter for some intermediate debt ranges. Unlike the stability result, this discrepancy *is* driven by the commitment assumptions. In particular, they assume that the sovereign takes q_t as given and cannot choose the best if multiple prices are available to him at time t. The 'point of no return' arises when there are some intermediate debt levels with two possible equilibrium prices, but higher debt levels imply a unique price.

In my set-up, the sovereign can always choose the highest price when multiple prices are available. Thus, by construction the solution is always unique in their sense of multiplicity: In that intermediate range, the sovereign can and would choose the better price. Thus, the multiplicity is not quite the same across the two models: A multiplicity of equilibria in my environment implies two distinct demand schedules, $q(b_{t+1})$, rather than just two distinct prices. Since the demand for debt does not depend on current debt levels but only future ones, the price does not depend on current debt levels and thus the multiplicity pervades for any initial debt level.

The counterfactual predictions regarding debt levels are another practical difference. In their counterfactual, for instance, both debt levels and spreads would be substantially lower; in this model, only spreads would be substantially lower. One could think of their model's Eurozone crisis counterfactual as the absence of a crisis, in which debt levels hover around pre-crisis levels; whereas this model would predict a counterfactual more like the US, in which debt levels still exploded but yields remained low and spreads near zero.

6 Conclusion

In this paper, I demonstrated that long-run market can influence markets for long-term sovereign debt via a multiplicity of steady states along the Lifetime-Laffer Curve. This multiplicity is ordered, which implies that intervention via liquidity provision to coordinate beliefs is always feasible and desirable, *regardless* of the initial debt level.

Applying this framework to the Eurozone, I established that these coordination failures could explain a potentially large fraction of Portuguese spreads during the crisis. I further showed during these crises that indeterminacy can allow even more room for sentiments to influence market outcomes, as expectations regarding the speed of convergence to the bad limiting outcome can be self-fulfilling.

There is still much room for further work. In particular, a method of identifying such sentiment-driven build-ups in the data would prove quite useful in verifying the applicability of the model. An investigation of the potential for this multiplicity in other types of financial markets suffering from similar frictions, such as the market for commercial paper or municipal debt, could also be in order.

Appendices

A Proofs of Theoretical Results

A.1 Proof of Lemma 1

The equilibrium conditions that define the system must satisfy the following

$$
\begin{aligned}
s(b_t) + &q(a(b_t))[a(b_t) - (1-\lambda)b_t] = \lambda b_t \\
q(b_{t+1}) = &\frac{1}{1+r}\left[[1 - g(b_{t+1})] \times [\lambda + (1-\lambda)q(a(b_{t+1}))]\right]
\end{aligned}
$$

where $b_{t+1} = a(b_t)$. Since $s(\cdot)$ and $g(\cdot)$ are continuous functions, this implies that $a(\cdot)$ and $q(\cdot)$ are also continuous functions.

Now, note that since $a : [0, b_u] \to [0, b_u]$, we have necessarily that $a(0) \geq 0$. Further, since $\lambda b_u - s(b_u) > 0$ but $q(b_u) = 0$, it must necessarily be the case that $a(b_u) < b_u$. Thus, the policy function is an increasing and continuous function that necessarily crosses the 45-degree line. Thus, any Monotone Markov Perfect Equilibrium converges to a steady state.

It could be, however, that the steady state is different from above and below: $a(\cdot)$ may cross the 45-degree line several times in its trajectory across its domain, which would imply that there are several steady states and that it converges to a higher steady state from above than below. I now provide conditions under which this cannot happen.

To go about this, I will show using the implicit function theorem that at any steady state, $\bar{b}$, it must be that $a'(\bar{b}) < 1$. Thus, the steady states neither zig-zag across the 45-degree line nor lie on top of it, since both of these would require $a'(\bar{b}) \geq 1$ for some $\bar{b}$.

I start by totally differentiating the budget constraint with respect to b_t:

$$s'(b_t) + q'(a(b_t))a'(b_t)[a(b_t) - (1-\lambda)b_t] + q(a(b_t))[a'(b_t) - (1-\lambda)] = \lambda$$

Notice that one can get rid of the price derivate by totally differentiating the pricing expression:

$$q'(a(b_t))a'(b_t) = \frac{1}{1+r}\Big[-g'(a(b_t))a'(b_t)[\lambda + (1-\lambda)q(a(a(b_t)))]$$
$$+\ [1-g(a(b_t))](1-\lambda)q'(a(a(b_t)))a'(a(b_t))a'(b_t)\Big]$$

We are only interested in analyzing these expressions at potential steady state, so the expression simplifies to

$$q'(\bar{b})a'(\bar{b}) = \frac{1}{1+r}\Big[-g'(\bar{b})a'(\bar{b})[\lambda + (1-\lambda)q(\bar{b})] + [1-g(\bar{b})](1-\lambda)q'(\bar{b})a'(\bar{b})^2\Big]$$
$$q'(\bar{b})a'(\bar{b}) = \frac{-\frac{1}{1+r}g'(\bar{b})a'(\bar{b})[\lambda + (1-\lambda)q(\bar{b})]}{1-\frac{1}{1+r}[1-g(\bar{b})](1-\lambda)a'(\bar{b})}$$
$$\implies q'(\bar{b})a'(\bar{b}) = \frac{-g'(\bar{b})a'(\bar{b})[\lambda + (1-\lambda)q(\bar{b})]}{1+r-[1-g(\bar{b})](1-\lambda)a'(\bar{b})} \quad \text{(A.1)}$$

Again, restricting attention to steady states, one can plug this back into our differentiated BC to get

$$s'(\bar{b}) - \frac{g'(\bar{b})a'(\bar{b})[\lambda + (1-\lambda)q(\bar{b})]}{1+r-[1-g(\bar{b})](1-\lambda)a'(\bar{b})}\lambda\bar{b} + q(\bar{b})[a'(\bar{b}) - (1-\lambda)] = \lambda \quad \text{(A.2)}$$
$$\implies -g'(\bar{b})[\lambda + (1-\lambda)q(\bar{b})]\lambda\bar{b}a'(\bar{b}) =$$
$$\Big[\lambda - s'(\bar{b}) - q(\bar{b})[a'(\bar{b}) - 1 + \lambda]\Big]\Big[1+r-[1-g(\bar{b})](1-\lambda)a'(\bar{b})\Big]$$

which can be rearranged to form the following expression, which is quadratic in $a'(\bar{b})$:

$$\begin{aligned} 0 = &\Big[q(\bar{b})[1-g(\bar{b})][1-\lambda]\Big]a'(\bar{b})^2 \\ &+\Big[g'(\bar{b})[\lambda+(1-\lambda)q(\bar{b})]\lambda\bar{b} - q(\bar{b})(1+r) - \\ &[1-g(\bar{b})][1-\lambda] - \Big(\lambda - s'(\bar{b}) + q(\bar{b})(1-\lambda)\Big)\Big]a'(\bar{b}) \\ &+ (1+r)\Big(\lambda - s'(\bar{b}) + q(\bar{b})(1-\lambda)\Big) \end{aligned} \tag{A.3}$$

Notice that the first term is positive, so it's U-shaped.

Notice next that evaluating Equation A.1 for $a'(\bar{b})$ as it approaches $\frac{1+r}{[1-g(\bar{b})][1-\lambda]}$ implies that it approaches minus infinity from below and plus infinity from above. Thus, in this neighborhood, the equality in Equation A.2 becomes a strict inequality. Carrying this inequality (from either side) through tells us that Equation A.3 evaluated at $\frac{1+r}{[1-g(\bar{b})][1-\lambda]}$ is less than zero.

Since our quadratic is U-shaped and we have identified a point below zero, we know that one root must lie to the left and one to the right. If $a'(\bar{b}) > \frac{1+r}{[1-g(\bar{b})][1-\lambda]}$, then Equation A.1 tells us that $q(\cdot)$ would be a strictly increasing function, which cannot be true since the policy function is increasing by definition. To see this, note that we can expand pricing Recursion 2 to be

$$q(b_{t+1}) = \frac{\lambda}{1-\lambda} \times \sum_{s=t}^{\infty} \left(\frac{1-\lambda}{1+r}\right)^{s-t+1} \Pi_{k=1}^{s-t+1}[1 - g(a^k(b_{t+1}))]$$

where $a^k(\cdot)$ is the kth application of the function $a(\cdot)$. Note that since $g(\cdot)$ and $a(\cdot)$ are both increasing, it must be that $q(\cdot)$ is decreasing.

Further, it cannot be that $a'(\bar{b}) < 0$ since the equilibrium is monotone. Thus, it must be the case that $a'(\bar{b}) \in \left[0, \frac{1+r}{[1-g(\bar{b})][1-\lambda]}\right]$. One is necessarily in this interval.

I can ensure further that this root is always less than unity provided that the quadratic expression remains negative when $a'(\bar{b}) = 1$. This implies the following restriction, which we arrive at by evaluating Equation A.3 at $a'(\bar{b}) = 1$:

$$r\Bigg(\lambda - s'(\bar{b}) + q(\bar{b})(1-\lambda)\Bigg) - q(\bar{b})[(1+r) - (1-g(\bar{b}))(1-\lambda)] -$$
$$[1-g(\bar{b})][1-\lambda] + g'(\bar{b})[\lambda + (1-\lambda)q(\bar{b})]\lambda\bar{b} < 0$$

Substituting in Equations 3 and 4, which must be satisfied in any Monotone Markov Perfect Equilibrium steady state, we arrive at the Global Convergence Condition given in Proposition 1.

A.2 Proof of Lemma 4

Notice that the Jacobian elements evaluated at the candidate solution are given below:

$$
\begin{aligned}
J_{11}(\bar{x}) =&(1-\lambda) + \frac{\lambda - s'(\bar{b})}{\bar{q}} \\
J_{12}(\bar{x}) =&-\frac{1}{\bar{q}^2}\left[\lambda \bar{b} - s(\bar{b})\right] \\
J_{21}(\bar{x}) =&\frac{\bar{q}(1+r)g'(\bar{b})}{[1-\lambda][1-g(\bar{b})]^2} J_{11}(\bar{x}) \\
J_{22}(\bar{x}) =&\frac{1+r}{[1-\lambda][1-g(\bar{b})]} + \frac{\bar{q}(1+r)g'(\bar{b})}{[1-\lambda][1-g(\bar{b})]^2} J_{12}(\bar{x})
\end{aligned}
$$

Notice that one can write the determinant as follows:

$$
\begin{aligned}
\det(J(\bar{x})) = J_{11}(\bar{x}) \times & \underbrace{\left(\frac{1+r}{[1-\lambda][1-g(\bar{b})]} + \frac{\bar{q}(1+r)g'(\bar{b})}{[1-\lambda][1-g(\bar{b})]^2} J_{12}(\bar{x})\right)}_{J_{22}(\bar{x})} \\
& - \underbrace{\frac{\bar{q}(1+r)g'(\bar{b})}{[1-\lambda][1-g(\bar{b})]^2} J_{11}(\bar{x})}_{J_{21}(\bar{x})} \times J_{12}(\bar{x}) \\
\rightarrow \det(J(\bar{x})) = J_{11}(\bar{x}) \times & \left(\frac{1+r}{[1-\lambda][1-g(\bar{b})]}\right)
\end{aligned}
$$

The latter term in this expression is always positive, and thus the determinant has the same sign as $J_{11}(\bar{x})$. By solving out for $\det(J - \hat{\lambda} I) = 0$, we arrive at a quadratic term with solutions:

$$\hat{\lambda}_1 = \frac{J_{11}(\bar{x}) + J_{22}(\bar{x}) + \sqrt{[J_{11}(\bar{x}) + J_{22}(\bar{x})]^2 - 4\det(J(\bar{x}))}}{2}$$
$$\hat{\lambda}_2 = \frac{J_{11}(\bar{x}) + J_{22}(\bar{x}) - \sqrt{[J_{11}(\bar{x}) + J_{22}(\bar{x})]^2 - 4\det(J(\bar{x}))}}{2}$$

Notice that this result is only valid when $\lambda \in (0,1)$. When $\lambda = 1$, the determinant is undefined. Thus, if $\lambda \in (0,1)$, then the characteristic roots of a candidate solution to the Lifetime-Laffer Curve, $(\bar{q}, \bar{b})$, are given by:

$$\hat{\lambda}_1 = \frac{B + \sqrt{B^2 - 4D}}{2}$$
$$\hat{\lambda}_2 = \frac{B - \sqrt{B^2 - 4D}}{2}$$

where

$$B = \underbrace{(1-\lambda) + \frac{\lambda - s'(\bar{b})}{\bar{q}}}_{X} + \underbrace{\frac{1+r}{[1-\lambda][1-g(\bar{b})]}}_{Y} - \underbrace{\frac{(1+r)g'(\bar{b})\lambda\bar{b}}{[1-\lambda][1-g(\bar{b})]^2}}_{Z}$$
$$D = \underbrace{\left((1-\lambda) + \frac{\lambda - s'(\bar{b})}{\bar{q}}\right)}_{X} \times \underbrace{\left(\frac{1+r}{[1-\lambda][1-g(\bar{b})]}\right)}_{Y}$$

These roots will be real if and only if $B^2 \geq 4D$. This will obviously be true when under Condition 1 of Lemma 4, since this implies that $D \leq 0$ whereas it is necessarily the case that $B^2 \geq 0$.

When Condition 1 does not hold the result can still be obtained. Notice that using the notation X, Y, and Z, requiring $B^2 \geq 4D$ is equivalent to saying

$$\begin{aligned} &(X + Y - Z)^2 \geq 4XY \\ \implies &\frac{X + Y}{2} \geq \sqrt{XY} + \frac{Z}{2} \end{aligned} \tag{A.4}$$

We know from that Arithmetic-Mean/Geometric-Mean inequality that if X and Y are positive then $\frac{X+Y}{2} \geq \sqrt{XY}$, with the inequality being strict whenever $X \neq Y$. It must be that $Y \geq 0$, and the first condition in the proposition is simply that $X \geq 0$.

The second condition simply imposes that Z, which is positive, is sufficiently small that Inequality A.4 holds. This is the formal statement of Condition 2.

A.3 Proof of Lemma 5

Theorem 6.1 in Stokey et al. (1989) states that the dimensionality of the stable manifold will be the same as the number of characteristic roots whose real values are less than unity in absolute value.

I continue with the notation from the proof of Lemma 4. First, I establish that $B \geq 0$. This will be true if if the root is real. To see this, suppose that $B < 0$. If this were the case, then it must be that $Z > X + Y$. But this would imply that $\frac{1}{2}(X + Y - Z) < 0$, which contradicts Inequality A.4, which requires that it be larger than $\sqrt{XY} \geq 0$. Thus, it must be the case that $B \geq 0$.

Since we also know if the root is real that $B^2 \geq 4D$, we will have that $B - \sqrt{B^2 - 4D} > 0$. This implies that both roots are not only real, but also positive i.e.

$$0 < \frac{B - \sqrt{B^2 - 4D}}{2} \leq \frac{B + \sqrt{B^2 - 4D}}{2}$$

The solution will thus be stable provided that $\frac{B-\sqrt{B^2-4D}}{2} < 1$, which will be true provided that $B < 2$, which can be re-written as the condition in Lemma 5.

A.4 Proof of Theorem 2

I will show this result via Tarski's Fixed Point Theorem. I restrict attention to real-valued, increasing functions that map the domain $[0, b_u]$ into itself. Call this set $\mathcal{A}$. I define partial ordering over these functions, $\geq$, as follows: If $a_1, a_0 \in \mathcal{A}$, then

$$a_1 \geq a_0 \iff a_1(b_t) \geq a_0(b_t)\ \forall b_t \in [0, b_u]$$

It is easy to see that $(\mathcal{A}, \geq)$ is a complete lattice. The supremum and infinum of any subset of $\mathcal{A}$ must necessarily exist in $\mathcal{A}$ since it is bounded and closed.

We now define an operator, $T : \mathcal{A} \to \mathcal{A}$, as follows. Let $a_i \in \mathcal{A}$. Define a real-valued function q_i via the pricing recursion:

$$q_i(b_{t+1}) = \frac{[1 - g(b_{t+1})] \times [\lambda + (1 - \lambda) q_i(a_i(b_{t+1}))]}{1 + r}$$

This is a contraction for any positive r and thus there is a unique pricing schedule, q_i. It need not be continuous, since I never restricted a_i to be continuous. However, it will be decreasing. To see this, note that one can expand the pricing recursion as follows:

$$q_i(b_{t+1}) = \frac{\lambda}{1 - \lambda} \times \sum_{s=t}^{\infty} \left(\frac{1 - \lambda}{1 + r} \right)^{s-t+1} \Pi_{k=1}^{s-t+1} [1 - g(a_i^k(b_{t+1}))]$$

where $a_i^k(b_{t+1})$ is the kth iteration of the policy function on the initial point b_{t+1}. Since $a_i(\cdot)$ is increasing, it follows trivially that $q_i(\cdot)$ is decreasing.

Having defined q_i, I now define Ta_i as follows. Let $B_i(b_t) = \{b_{t+1} | q_i(b_{t+1})[b_{t+1} -$

$(1-\lambda)b_t] = \lambda b_t - s(b_t)$ and $q_i(b_{t+1}) \geq q_{min}\}$, where $q_{min} = \frac{\sup_b s'(b)-\lambda}{1-\lambda}$.

$$(Ta_i)(b_t) = \begin{cases} b_u, & \text{if } B_i(b) \text{ is empty for some } b \leq b_t \\ \\ \inf B_i(b_t), & \text{otherwise} \end{cases} \tag{A.5}$$

This operator dictates that the updated policy function meet the budget constraint whenever it is feasible to do so *and* provided the price does not drop too low i.e. below $q_{min}(b_t)$.

This latter restriction is necessary for the following reason: Should the price drop too low, it is possible to lose monotonicity in the updated policy function. This restriction prevents this. Suppose that consider some $b_L < b_H$ such that $B_i(b_L)$ and $B_i(b_H)$ would be non-empty without this price restriction. We know that

$$q_i((Ta_i)(b_H))[(Ta_i)(b_H) - (1-\lambda)b_H] -$$
$$q_i((Ta_i)(b_L))[(Ta_i)(b_L) - (1-\lambda)b_L] = [\lambda b_H - s(b_H)] - [\lambda b_L - s(b_L)]$$

We now conjecture and verify that $(Ta_i)(\cdot)$ is increasing. If it is, then because $q_i(\cdot)$ is decreasing, we will have $q_i((Ta_i)(b_H)) \leq q_i((Ta_i)(b_L))$. This implies (together with the assumption of no buybacks)

$$q_i((Ta_i)(b_L))\Big([(Ta_i)(b_H) - (1-\lambda)b_H] - [(Ta_i)(b_L) - (1-\lambda)b_L]\Big) \geq$$

$$[\lambda - \sup_b s'(b)][b_H - b_L]$$

$$\implies (Ta_i)(b_H) - (Ta_i)(b_L) \geq [b_H - b_L]\left(\frac{\lambda - \sup_b s'(b)}{q_i((Ta_i)(b_L))} - (1-\lambda)\right)$$

This implies that (Ta_i) will be increasing provided the pricing restriction is satisfied, even in those cases in which revenue needs are decreasing in b_t.

I now show that T is a monotone map. Suppose that we have two policies in $\mathcal{A}$: $a_1 \geq a_0$. I first show that $q_1(b_{t+1}) \leq q_0(b_{t+1})$. To see this, note that one can expand the pricing recursion into an infinite sum, in which case:

$$\begin{aligned} q_1(b_{t+1}) &= \frac{\lambda}{1-\lambda} \times \sum_{s=t}^{\infty} \left(\frac{1-\lambda}{1+r}\right)^{s-t+1} \Pi_{k=1}^{s-t+1}[1 - g(a_1^k(b_{t+1}))] \\ &\leq \frac{\lambda}{1-\lambda} \times \sum_{s=t}^{\infty} \left(\frac{1-\lambda}{1+r}\right)^{s-t+1} \Pi_{k=1}^{s-t+1}[1 - g(a_0^k(b_{t+1}))] \\ &= q_0(b_{t+1}) \end{aligned}$$

where $a_i^k(\cdot)$ is the application of the function $a_i(\cdot)$ k times. The inequality follows since $a_1^k(b_{t+1}) \geq a_0^k(b_{t+1})$ since both are increasing functions and since $a_1 \geq a_0$.

Having established that the pricing schedules are ordered, there are essentially three cases that could arise:

1. Both $(Ta_1)(b_t)$ and $(Ta_0)(b_t)$ are feasible. In this case,

$$
\begin{aligned}
q_1(b^1_{t+1})[b^1_{t+1} - (1-\lambda)b_t] &\geq \lambda b_t - s(b_t) \geq 0 \\
\rightarrow q_0(b^1_{t+1})[b^1_{t+1} - (1-\lambda)b_t] &\geq \lambda b_t - s(b_t) \\
\rightarrow b^0_{t+1} &\leq b^1_{t+1}
\end{aligned}
$$

   In this case, we know that that q_0 is always bigger than q_1 for any b_{t+1}, but there are two possible cases. Either $q_0(b^1_{t+1})[b^1_{t+1} - (1-\lambda)b_t]$ is an increasing function, in which case the last inequality must hold since b^0_{t+1} is the infinum of that set; or $q_0(b^1_{t+1})[b^1_{t+1} - (1-\lambda)b_t]$ is a decreasing function, in which case we can attain the same revenue by shifting to the good, increasing side of the Laffer curve, $\hat{b}^1_{t+1} \leq b^1_{t+1}$. From here, I can apply the same argument as before to argue for the last line since $b^0_{t+1} \leq \hat{b}^1_{t+1}$.

2. Both $(Ta_1)(b_t)$ and $(Ta_0)(b_t)$ are infeasible. In this case it must be that $(Ta_1)(b_t) = (Ta_0)(b_t) = b_u$.

3. Exactly one of $(Ta_1)(b_t)$ and $(Ta_0)(b_t)$ is feasible. In this case, we know that the feasible one must be $(Ta_0)(b_t)$, since the price is uniformly higher in this case. By the fact that $T : \mathcal{A} \rightarrow \mathcal{A}$, we will have that

$$(Ta_0)(b_t) \leq b_u = (Ta_1)(b_t).$$

In examining all three cases, it is clear that $Ta_1 \geq Ta_0$, and thus the operator is a monotone self-map. Applying Tarski's fixed point theorem tells us that the set of fixed points is a complete lattice, which implies that at least one exists and that they are all ordered according to $\geq$.

Now this does not tell us that a Markov Perfect Equilibrium exists, since a Markov Perfect Equilibrium will be a fixed point in which Ta exhibits feasibility for any b_t in Equation A.5. If there are multiple fixed points that are fully feasible in this sense, however, then the ordering result from Tarski's Fixed Point Theorem applies and we will have Theorem 2.

B Solution Algorithm

We know from Theorem 2 that iterative algorithms will do well to uncover the lowest-debt equilibrium, but not necessarily any high-debt equilibrium. In fact, iterative procedures from arbitrary initial guesses almost never uncover the high-debt equilibrium in the presence of multiplicity. The monotonicity in the functional operator does not actually help us here either, since it tends to go either to the low-debt equilibrium or the degenerate solution $a(b_t) = b_u$ for all b_t. While this latter point is not an equilibrium, it is a fixed point of the functional operator.

This does not mean that a high-debt equilibrium does not exist; only that it's trickier to find. To accomplish this will require an alternative method, which I propose here. The idea is to apply a shooting algorithm on the original dynamical system: Given b_t, we choose some q_t; the budget constraint then yields some b_{t+1}; q_{t+1} can then be chosen to satisfy the no-arbitrage condition of the lenders; the system is then ratcheted forward until a steady state is hopefully reached. Once there, we can use the implied trajectory to construct a guess for the policy and demand schedules.

The problem with such an algorithm is that in its current form it treats the sovereign as a price-taker, when in our equilibrium concept the sovereign can always choose the best price by choosing the smallest of potential solutions for b_{t+1}. To try to get at this additional restriction, I tweak the dynamical system in such a way as to obviate this contemporary multiplicity of solutions; in doing so, I can apply the shooting algorithm described above and ratchet toward a solution that assumes a variation on such commitment powers. The solution to this approximating model can then be used as an initial guess in an iterative procedure on the original model. In practice, this algorithm tends to perform quite well in its ability to uncover the multiplicity of equilibria.

B.1 The Approximating Model

I begin by describing the approximating model.

$$\frac{s(b_t)}{\Delta} + q_t \times \left[b_{t+1/\Delta} - \left(1 - \frac{\lambda}{\Delta}\right) b_t\right] = \frac{\lambda}{\Delta} b_t \tag{B.1}$$

$$q_t = \frac{1}{1 + \frac{r}{\Delta}} \times \left[\left[1 - g(b_{t+1/\Delta})\right] \times \left(\frac{\lambda}{\Delta} + \left(1 - \frac{\lambda}{\Delta}\right) q_{t+1/\Delta}\right) + g(b_{t+1/\Delta}) \times \left(0 + \left(1 - \frac{1}{\Delta}\right) q_{t+1/\Delta}\right)\right] \tag{B.2}$$

This approximating model splits the original model into subperiods of length $1/\Delta$ in such a way that the Lifetime-Laffer Curve is preserved exactly i.e. if one were to compute the set of steady states in this model, they would be exactly the same as in the original model outlined in Section 2. In fact, that model is simply a special case of this one when $\Delta = 1$.

Notice from Equation B.2 that to do this requires some recovery following a default i.e. in the event of a default, current coupon and maturity obligations are not met, but they are just as likely to be met in future periods as if the sovereign had not defaulted. Further the size of current coupon and maturity obligations shrink with Δ, while weight placed on their continuation value increases with Δ.

The benefit of this approximation is that for large Δ it lops off the Pareto-Dominated right-hand solution of the contemporaneous Laffer curve given by Figure 1, thus guaranteeing the contemporaneous commitment. To see why, notice that revenue needs in any subperiod are equal to $\frac{\lambda b_t - s(b_t)}{\Delta}$, which is decreasing in Δ. Simultaneously, recovery values following default are increasing in Δ, which bound the decreasing portion of the revenue function away from zero. Thus, as Δ increases, the black dotted line governing revenue needs in Figure 1 falls to zero, while the right-hand side of the revenue function pulls up away from zero. When Δ gets large enough, eventually only one solution will prevail, which is the low-debt solution in which we're interested.

B.2 Algorithm

I begin by re-writing Equations B.1 and B.2 as follows:

$$b_{t+1/\Delta} = \left(1 - \frac{\lambda}{\Delta}\right) b_t + \frac{\frac{\lambda}{\Delta} b_t - \frac{1}{\Delta} s(b_t)}{q_t}$$

$$q_{t+1/\Delta} = \frac{q_t \left(1 + \frac{r}{\Delta}\right) - \left[1 - g(b_{t+1/\Delta})\right] \frac{\lambda}{\Delta}}{\left(1 - \frac{\lambda}{\Delta}\right) - \frac{g(b_{t+1/\Delta})}{\Delta}(1 - \lambda)}$$

How can this deconstruction be interpreted? Given some initial q_t, the budget constraint will give the required $b_{t+1/\Delta}$ required. Once we know $b_{t+1/\Delta}$, we must then rationalize the chosen q_t. This is done by adjusting the return in the future via $q_{t+1/\Delta}$ until the lenders' break-even condition is met.

This approach is used by Lorenzoni and Werning (2014), who explore a similar model in continuous time. The problem is that this method cannot guarantee contemporaneous commitment to debt issuance. With the approximating model, however, we know that only one solution exists for large Δ and that it is the 'good' one.

Thus, to solve the model, we simply apply a shooting algorithm to the approximating model for sufficiently large Δ. In practice, setting $\Delta = 20$ tends to work very well.

We consider two starting points, $b_0 = 0$ and $b_0 = b_u$. In each case, we employ a shooting algorithm over q_0, ratcheting the iterations forward. If they converge in a monotone way to a steady state, then we have found a solution to the approximating model for that Δ.

Our solution will consist of two sets of points: A set $\{(q_0, b_0), (q_{1/\Delta}, b_{1/\Delta}), (q_{2/\Delta}, b_{2/\Delta}), \dots\}$, all of which imply $b_t \geq \bar{b}$, and a set $\{(\hat{q}_0, \hat{b}_0), (\hat{q}_{1/\Delta}, \hat{b}_{1/\Delta}), (\hat{q}_{2/\Delta}, \hat{b}_{2/\Delta}), \dots\}$, all of which imply $\hat{b}_t \leq \bar{b}$. If the trajectories are monotone and contractionary around the steady state, then we can construct a point-wise estimates of the policy function and the demand schedule as follows:

$$a(b_t) = \begin{cases} b_{t+1/\Delta}, & b_t > \bar{b} \\ \hat{b}_{t+1/\Delta}, & b_t \leq \bar{b} \end{cases}$$

$$q(b_{t+1/\Delta}) = \begin{cases} q_t, & b_{t+1/\Delta} > \bar{b} \\ \hat{q}_t, & b_{t+1/\Delta} \leq \bar{b} \end{cases}$$

One could interpolate on these point-wise estimates to construct an initial guess for an iterative algorithm.

Denote the solution found using the shooting algorithm above as $a_\Delta(b_t)$ and $q_\Delta(b_{t+1/\Delta})$. I will solve the general model using an iterative algorithm. Let $a^i(b_t)$ and $q^i(b_{t+1})$ denote the estimate of the policy and price functions at iteration i.

I begin by setting $a^0(b_t) = a_\Delta(b_t)$ and $q^0(b_{t+1}) = q_\Delta(b_{t+1/\Delta})$. I then form a grid, $\mathcal{B}$ over the domain of b_t and proceed as follows:

1. For every $b_t \in \mathcal{B}$, solve $q^i(b_{t+1})[b_{t+1} - (1-\lambda)b_t] = \lambda b_t - s(b_t)$ for b_{t+1}.

   There will necessarily be multiple solutions for $i \geq 1$. Define

$$a^{i+1}(b_t) =$$

$$\min_{b' \in \mathcal{B}} \left(b' - \inf\{b_{t+1} | q^i(b_{t+1})[b_{t+1} - (1-\lambda)b_t] = \lambda b_t - s(b_t)\}\right)^2$$

2. For every $b_{t+1} \in \mathcal{B}$, define

$$q^{i+1}(b_{t+1}) = \frac{[1 - g(b_{t+1})] \times [\lambda + (1-\lambda) q^i(a^{i+1}(b_{t+1}))]}{1+r}$$

3. Continue until $\sup_{b' \in \mathcal{B}} |q^i(b') - q^{i+1}(b')| < \epsilon$ for some small ϵ.

This process generally converges to a high-debt equilibrium, though it is not guaranteed to do so since it is not a contraction. If one attempts this process from an arbitrary initial guess not tailored by our shooting algorithm, it never appears to succeed in finding a high-debt equilibrium.

C Assumptions in the Calibrated Example

I show now how the relevant assumptions are satisfied in the calibrated example

- **Global Convergence Condition:** There are two potential steady states. For each, the global convergence condition holds, with the exact figures for each evaluated below.
  1. $r + \lambda b_L g'(b_L)] \left[1 - \frac{s(b_L)[1-\lambda]}{\lambda b_L}\right] = 0.0131 < 0.9972 = rs'(b_L) + [1 - g(b_L)]$
  2. $r + \lambda b_H g'(b_H)] \left[1 - \frac{s(b_H)[1-\lambda]}{\lambda b_H}\right] = 0.0613 < 0.9952 = rs'(b_H) + [1 - g(b_H)]$
- **Assumption 1:** $\lim_{b \to 0} s(b)/b = -\infty < \frac{r}{r+\lambda}$
- **Assumption 2:** $s(b_u)/b_u = 0.0431 < 0.0437 = \lambda$

- **Assumption 3:** Any $\hat{b} \in (b_L, b_H)$ will satisfy this property (see Figure 2).

- **Assumption 4:** Condition 2 holds in each case. In the notation of Equation A.4, for each solution along the Lifetime-Laffer Curve, we will require $Z \leq 2\left[\frac{X+Y}{2} - \sqrt{XY}\right]$

  1. b_L: $Z(bL) = 0.0132 \leq 0.1365 = 2\left[\frac{X(bL)+Y(bL)}{2} - \sqrt{X(bL)Y(bL)}\right]$
  2. b_H: $Z(bH) = 0.1245 \leq 0.5712 = 2\left[\frac{X(bH)+Y(bH)}{2} - \sqrt{X(bH)Y(bH)}\right]$

- **Assumption 5:** There are two solutions along the Lifetime-Laffer Curve. Each sets a lower bound on the surplus function that is satisfied

  1. $s'(b_H) = 0.5 > 0.0780 = \lambda - \frac{\lambda[1-g(b_L)]}{r+g(b_L)+\lambda[1-g(b_L)]}\left(1 + \frac{1+r}{[1-\lambda][1-g(b_L)]}\left[\lambda b_L \frac{g'(b_L)}{1-g(b_L)} - 1\right]\right)$
  2. $s'(b_L) = 0.5 > 0.0244 = \lambda - \frac{\lambda[1-g(b_H)]}{r+g(b_H)+\lambda[1-g(b_H)]}\left(1 + \frac{1+r}{[1-\lambda][1-g(b_H)]}\left[\lambda b_H \frac{g'(b_H)}{1-g(b_H)} - 1\right]\right)$

- **Assumption 6:** Revenue needs are always positive

  - $\lambda b_t - s(b_t)$ is a decreasing function since $s(\cdot) = \kappa_0 + \kappa_1 b_t$ and $\kappa_1 > \lambda$
  - $\lambda b_u - s(b_u) = 0.0006$

The characteristic roots of the two steady states are given below:

$$\hat{\lambda}_{L,1} = 1.0299$$

$$\hat{\lambda}_{L,2} = 0.4405$$

$$\hat{\lambda}_{H,1} = 0.9513$$

$$\hat{\lambda}_{H,2} = 0.0942$$

From this, it is clear that the high-debt solution is stable in the sense that any nearby point will eventually converge to (q_H, b_H). The low-debt solution, however, only has a one-dimensional stable manifold i.e. a saddle path. Since the dynamical system does not explicitly include an optimality condition, which in this case serves the purpose of selecting a solution, a continuum of local solutions surrounding b_H need not correspond to a multiple equilibria traveling to b_H. In this case, however, it appears to (see Result 1).

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