International Macroeconomics Supplemental Lecture: Solving Dynamic Systems Graphically

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Dynamic Systems

- Many economic models yield law of motion for two variables over time: (x_t, y_t)

$$x_{t+1} = f(x_t, y_t)$$

$$y_{t+1} = g(x_t, y_t)$$

- f and g could be linear or nonlinear

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- f and g could be linear or nonlinear
- Want some notion of how these variables jointly evolve and to where they travel
  - Can evaluate iteratively to find out from some initial conditions

$$x_{t+2} = f(x_{t+1}, y_{t+1})$$

$$y_{t+2} = g(x_{t+1}, y_{t+1})$$

- Starting from initial (x_0, y_0), can iterate to determine trajectory as $t \to \infty$

Steady States

- First solve model for differences in variables

$$\Delta x_{t+1} = x_{t+1} - x_t = f(x_t, y_t) - x_t$$
$$\Delta y_{t+1} = y_{t+1} - y_t = g(x_t, y_t) - y_t$$

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- Solve for steady state lines: $\{(\hat{x}, \hat{y})\} \rightarrow \Delta x = \Delta y = 0$

$$\hat{x} = f(\hat{x}, \hat{y})$$

or

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- System steady states satisfies both simultaneously

$$\bar{x} = f(\bar{x}, \bar{y})$$

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Trajectories

- Trajectories can be backed out in relation:

$$x_t < f(x_t, y_t) \rightarrow \Delta x_{t+1} < 0$$

$$x_t > f(x_t, y_t) \rightarrow \Delta x_{t+1} > 0$$

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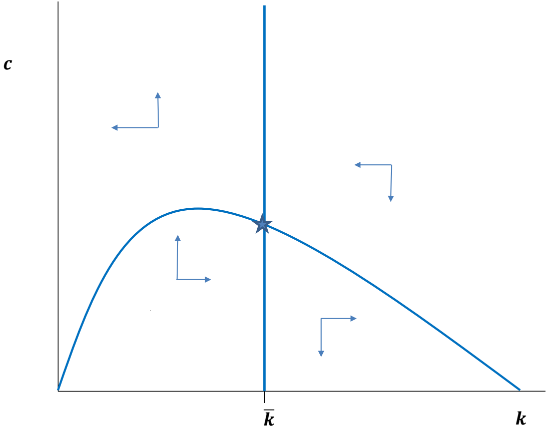
- Example: Neoclassical Growth Model (k_t, c_t)
  1. Resource Constraint: $k_{t+1} = f(k_t) + k_t - c_t$
  2. Euler Equation: $c_{t+1} = \beta^\sigma [1 + f'(k_{t+1})]^\sigma c_t$

NCG Example: SS Lines and Trajectories



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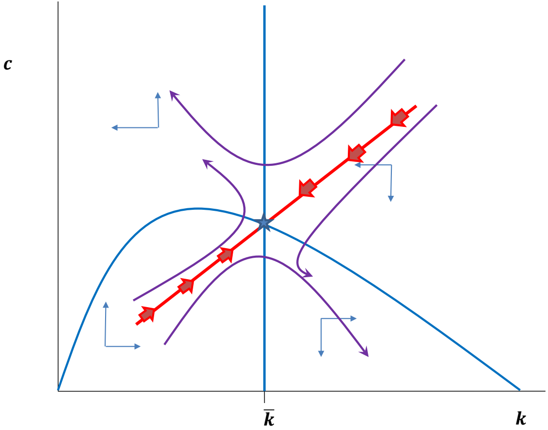
  - Determine other variable by placing it on saddle path to satisfy stability
  - Example: k_0 given; c_0 determined by saddle path

NCG Example: Saddle-Path Stability



NCG Example: Solution